5. GAUSSIAN BEAMS

5.1. Solution to the wave equation in Cartesian coordinates

- Recall the Helmholtz equation for a scalar field U in rectangular coordinates

$$\nabla^2 U(\mathbf{r},\omega) + \beta^2(\mathbf{r},\omega)U(\mathbf{r},\omega) = 0, \qquad\qquad (5.1)$$

- β is the wavenumber, defined as

$$\beta^2(\mathbf{r},\omega) = \omega^2 \mu\varepsilon(\mathbf{r},\omega) - i\omega\mu\sigma(\mathbf{r},\omega)$$

$$= n^2(\mathbf{r},\omega)\frac{\omega^2}{c^2} - i\omega\mu\sigma(\mathbf{r},\omega) \qquad (5.2)$$

- Assuming lossless medium $(\sigma = 0)$ and decoupling the vacuum contribution $(n = 1)$ from $\beta(\mathbf{r},\omega)$, we re-write Eq. 2 to explicitly show the driving term,

$$\nabla^2 U(\mathbf{r},\omega) + k_0^2 U(\mathbf{r},\omega) = -k_0^2 \left[n^2(\mathbf{r},\omega) - 1 \right] U(\mathbf{r},\omega), \ (5.3)$$

- $k_0 = \omega/c$.

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- Eq. 3 preserves the generality of the Helmholtz equation. Green's function, h, (an impulse response) is obtained by setting the driving term to a delta function,

$$\nabla^2 h(\mathbf{r},\omega) + k_0^2 h(\mathbf{r},\omega) = -\delta^{(3)}(\mathbf{r})$$
$$= -\delta(x)\delta(y)\delta(z) \qquad (5.4)$$

- To solve this equation, we take the Fourier transform with respect to $\mathbf{r}$,

$$-k^2 \tilde{h}(\mathbf{k},\omega) + k_0^2 \tilde{h}(\mathbf{k},\omega) = -1, \qquad (5.5)$$

- $k^2 = \mathbf{k} \cdot \mathbf{k}$.

- This equation breaks into three identical equations, for each spatial coordinate,

$$\tilde{h}(k_x,\omega) = \frac{1}{k_x^2 - k_0^2}$$
$$= \frac{1}{2k_{0x}}\left[\frac{1}{k_x - k_0} - \frac{1}{k_x + k_0}\right] \qquad (5.6)$$

- To calculate the Fourier transform of Eq. 6, we invoke the shift theorem and the Fourier transform of function $\frac{1}{k}$,

$$f(k-a) \rightarrow e^{iax} \cdot \tilde{f}(x)$$
$$\frac{1}{k} \rightarrow i\,\text{sign}(x) \qquad\qquad (5.7)$$

- Thus we obtain as the final solution

$$h(x,\omega) = \frac{1}{ik_0} \cdot e^{ik_0 x}, \quad x > 0 \qquad\qquad (5.8)$$

- The procedure applies to all three dimensions, such that the 3D solution reads

$$h(\mathbf{r},\omega) \propto e^{ik_0 \hat{\mathbf{k}} \cdot \mathbf{r}}, \qquad\qquad (5.9)$$

- $\hat{\mathbf{k}}$ is the unit vector, $\hat{\mathbf{k}} = \mathbf{k}/k$.

- Equation 9 describes the well known *plane wave* solution, which is characterized by the absence of amplitude modulation upon propagation. This is an infinitely broad wavefront that propagates along direction $\hat{\mathbf{k}}$ (Figure 1).
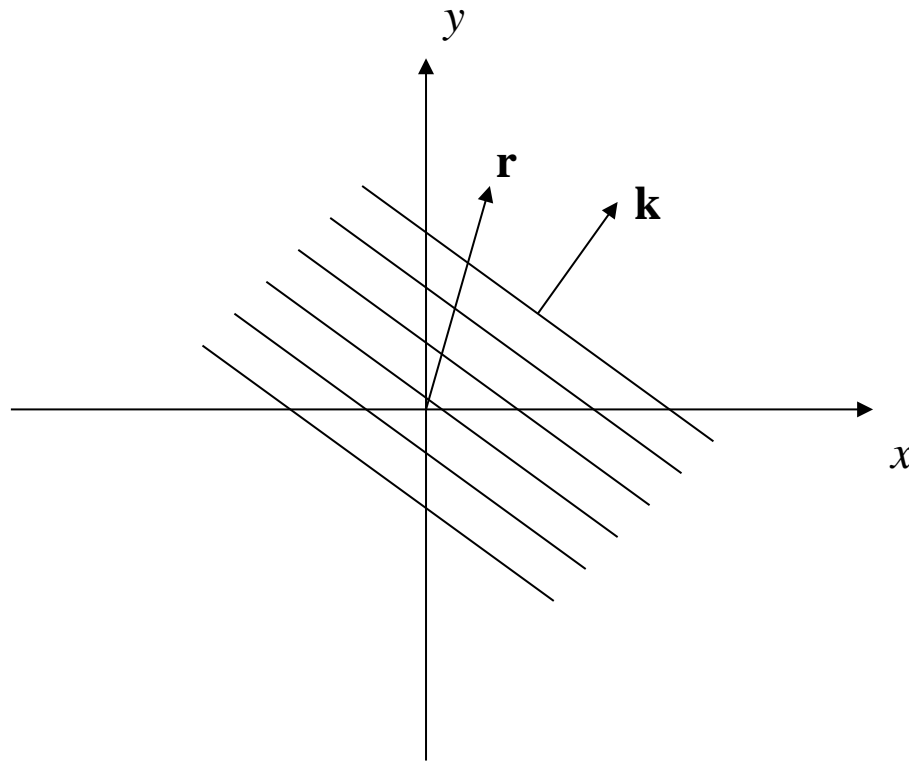
Figure 1. Plane wave.

5.2. Solution of the wave equation in spherical coordinates

- For spherical symmetry, such as emission from a point source in free space, the problem becomes one dimensional, with the radial coordinate as the only variable,

$$h(\mathbf{r}, \omega) = U(r, \omega)$$
$$\tilde{h}(\mathbf{k}, \omega) = \tilde{U}(k, \omega)$$
$$^{(3)}\delta(\mathbf{r}) = \frac{1}{2\pi r^2}^{(1)}\delta(r) \qquad\qquad (5.10)$$
$$^{(3)}\delta(\mathbf{r}) \to 1$$

- Recall that the Fourier transform pair is defined in this case as

$$h(r) \propto \int_0^\infty \tilde{h}(k) \cdot \mathrm{sinc}(kr) k^2 dk$$
$$\qquad\qquad (5.11)$$
$$\tilde{h}(k) \propto \int_0^\infty h(r) \cdot \mathrm{sinc}(kr) r^2 dr$$

- The Fourier properties of $^{(3)}\delta(r)$ and ∇^2 extend naturally to the spherically symmetric case as

$$^{(3)}\delta(r) \to 1$$
$$\nabla^2 \to -k^2$$

$$(5.12)$$

- Thus, by Fourier transforming Eq. 4, we obtain the frequency domain solution,

$$\tilde{h}(k,\omega) = \frac{1}{k^2 - k_0^{\,2}}$$

$$(5.13)$$

- Not surprisingly, the frequency domain solutions for the Cartesian and spherical coordinates (Eqs. 6 and 13, respectively) look quite similar, except that the former depends on one component of the wave vector and the latter on the modulus of the wave vector.

- The solution in the spatial domain becomes

$$h(r,\omega) = \int_0^\infty \frac{1}{k^2 - k_0^2} \cdot \frac{\sin(kr)}{kr} \cdot k^2 dk$$

$$= \frac{1}{2r} \int_0^\infty \left(\frac{1}{k - k_0} + \frac{1}{k + k_0} \right) \frac{e^{ikr} - e^{-ikr}}{2i} dk \quad (5.14)$$

$$= \frac{1}{4ir} \int_0^\infty \frac{e^{ikr}}{k - k_0} dk.$$

- We recognize in Eq. 15 the Fourier transform of a shifted $\frac{1}{k}$ function, which we encountered earlier (Eqs. 7). Evaluating this Fourier transform, we finally obtain Green's function for propagation from a point source,

$$h(r,\omega) \propto \frac{e^{ikr}}{r}, \quad r > 0 \qquad\qquad (5.15)$$

- This solution defines a (outgoing) spherical wave.

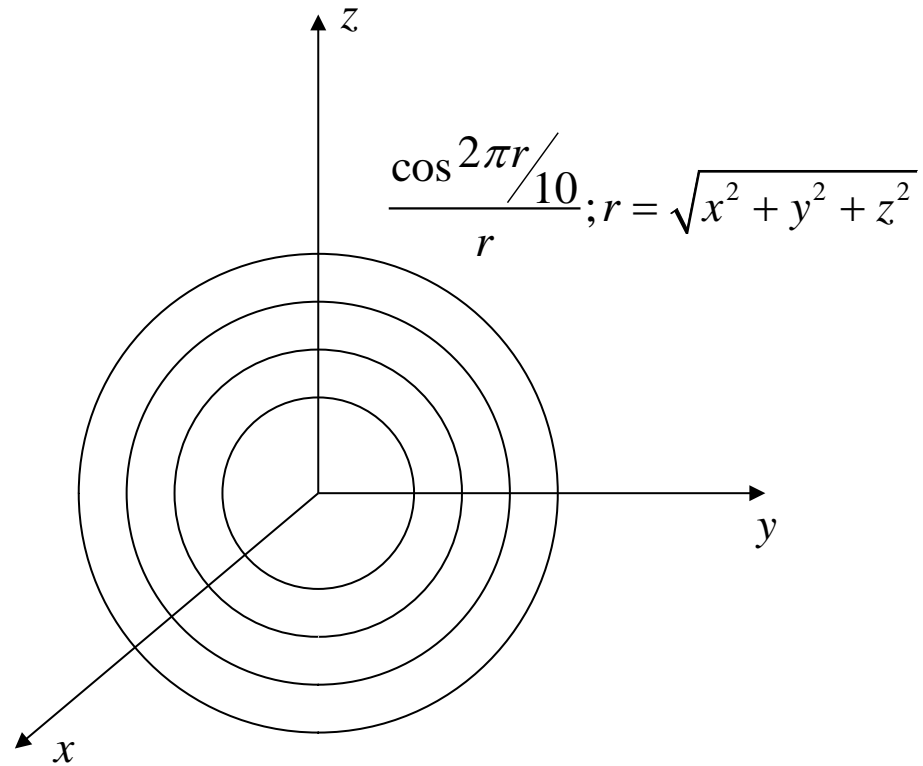$$\frac{\cos \frac{2\pi r}{10}}{r}; r = \sqrt{x^2 + y^2 + z^2}$$

Figure 5-2. Spherical wave

5.3. Solution of the wave equation in cylindrical coordinates

- The most common situation from an experimental point of view is light propagation in cylindrical coordinates with radial symmetry, the z coordinate defines the optical axes and the radial coordinate, r, defines the transverse coordinate.

- The Fourier pair in this case can be written as

$$h(r,z) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \tilde{h}(k_\perp, k_z) J_0(k_\perp \cdot r) \cdot e^{ik_z \cdot z} \cdot k_\perp \cdot dk_\perp dk_z$$

$$\tilde{h}(k_\perp, k_z) = \int_{-\infty}^{\infty} \int_{0}^{\infty} h(r,z) J_0(k_\perp \cdot r) \cdot e^{-ik_z \cdot z} \cdot r \cdot dr dz$$

(5.16)

- Due to the angular symmetry in both the spherical and cylindrical coordinates, the spatial frequency domain solution of the Helmholtz equation is identical in form with Eq. 13 and depends only on the modulus of the wave vector

$$\tilde{h}(k,\omega) = \frac{1}{k^2 - k_0^2}, \qquad\qquad (5.17)$$

$$k^2 = k_\perp^2 + k_z^2$$

- $\tilde{U}(k,\omega)$ was obtained from the procedure used in the spherically symmetric case, replacing the source term with $^{(3)}\delta(r,z)$ and using the property $\nabla^2 \to -k^2$ that holds for cylindrical coordinates with circular symmetry.

- We now Fourier transform Eq. 13 back to the spatial domain,

$$h(r,z) = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{1}{k_z^2 - \left(k_0^2 - k_\perp^2\right)} \cdot J_0\left(k_\perp \cdot r\right) \cdot e^{ik_z \cdot z} k_\perp dk_\perp dk_z \qquad (5.18)$$

- Eq. 18 can be interpreted as the Hankel transform (over $k_\perp$) of a 1D Fourier transform (over k_z). In evaluating the integral over k_z first, we recover the 1D case discussed in Section 5.1, the solution in Eq. 8, where we replace k_0 with $\sqrt{k_0^2 - k_\perp^2}$ and x with z.

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- The remaining integral over $k_\perp$ becomes

$$h(r,z) = \int_0^\infty \frac{e^{iqz}}{q} \cdot J_0\left(k_\perp \cdot r\right) \cdot k_\perp dk_\perp,$$
$$q = \sqrt{k_0^2 - k_\perp^2}$$

(5.19)

- Equation 19a can be regarded as the exact Green's function in cylindrical coordinates with circular symmetry. The integral can be evaluated analytically in the limit of low transverse frequencies, i.e. $k_\perp \ll k_0$.

- In this case we expand $\sqrt{k_0^2 - k_\perp^2}$ in Taylor series to obtain

$$\frac{e^{iqz}}{q} \simeq \frac{e^{ik_0 z\left[1 - \frac{k_\perp^2}{2k_0^2}\right]}}{k_0}.$$

(5.20)

- The amplitude term varies slowly with $k_\perp$, so the zeroth order term is sufficient, but the phase term is more sensitive, thus the second order term is necessary.

- Following this expansion, the Hankel transform remains to be calculated for a Gaussian phase function $e^{-\frac{iz}{2k_0}\cdot k_\perp^2}$. Recall that Hankel transforms of Gaussian functions can be easily calculated and follows a similar transformation rule with that of 1D functions, $e^{-\frac{ix^2}{2b^2}} \rightarrow b^2 e^{\frac{ib^2k^2}{2}}$.

- By combining Eqs. 20 and 19a, we obtain

$$h(r,z) \propto \frac{e^{ik_0 z}}{z} \cdot e^{\frac{ik_0}{2z}\cdot r^2}. \qquad\qquad (5.21)$$

- This solution is characterized by a slow attenuation with distance, $\propto \sqrt{z}$, a plane wave-like phase delay vs. z, and a Gaussian transverse phase, of width that also grows as $\sqrt{z}$.

5.4. Propagation of Gaussian beams

- Often, experiments involve *light beams*. A light beam can be defined as a distribution of field that fulfills the approximation in Eq. 20, is characterized by a dominant wave vector component, $k_z \gg k_x, k_y$. A beam is the spatial equivalent of *quasi-monochromatic light*, where the field is characterized by a dominant (temporal) frequency component.

- A *Gaussian beam*, such as that delivered by a single mode laser, has a field distribution described by a Gaussian function in the transverse coordinate. Green's function in Eq. 21 provides a way to propagate this type of field.

- Consider a Gaussian beam that at z=0 has the transverse distribution

$$U_0(r,0) = \frac{1}{w_0^2} e^{-\frac{r^2}{w_0^2}}, \qquad\qquad (5.22)$$

- w_0 defines the minimum beam waist.

- Upon propagating a distance z, the field becomes a convolution between U_0 and h,

$$U(r,z) = U_0(r,0) * h(r,z) \tag{5.23}$$

- We take the Fourier transform of Eq. 23, make use of the convolution theorem and use the approximation in Eq. 20 to obtain the frequency domain solution

$$\tilde{U}(k_\perp, z) = \frac{e^{ik_0 z}}{k_0} \cdot e^{-\frac{izk_\perp^2}{2k_0}} \cdot e^{-\frac{w_0^2 k_\perp^2}{4}} \tag{5.24}$$

- The spatial domain solution is obtained by taking the inverse Hankel transform of $\tilde{U}$,

$$U(r,z) = \int_0^\infty U(k_\perp, z) \cdot J_0(k_\perp r) k_\perp dk_\perp. \tag{5.25}$$

- After propagation over distance z, the field remains Gaussian in the transverse direction and has the form

$$U(r,z) = \frac{e^{ik_0 z}}{c^2(z)} e^{-\frac{r^2}{c^2(z)}}$$

$$c^2(z) = w_0^2 + i\frac{2z}{k_0}$$

(5.26)

- The solution has a particularly simple form, thanks to the properties of Gaussian functions and their transforms. All the information regarding the beam change due to propagation is contained in function c(z). This function can be rearranged

$$c^2(z) = w_0^2\left(1 + i\frac{z}{z_0}\right)$$

$$z_0 = \frac{k_0 w_0^2}{2} = \frac{\pi w_0^2}{\lambda}$$

(5.27)

- We can write explicitly the absolute value and phase of $c^2(z)$, express it in polar coordinates, as

$$c^2(z) = w^2(z) \cdot e^{i\psi(z)}, \qquad\qquad (5.28)$$

- where

$$w^2(z) = w_0^2 \left(1 + \frac{z^2}{z_0^2} \right)$$

$$\psi(z) = \arg\left(\frac{z}{z_0} \right) \qquad\qquad (5.29)$$

- The distance z_0 is known as the Rayleigh range, over which the beam waist increases by $\sqrt{2}$. We can re-write U(r, z) explicitly,

$$U(r,z) = \frac{1}{w^2(z)} \cdot e^{i\left[k_0 z + \frac{z}{z_0} \frac{r^2}{w^2(z)} - \psi(z) \right]} \cdot e^{-\frac{r^2}{w^2(z)}} \qquad\qquad (5.30)$$

$$= A(r,z) \cdot e^{i\phi(r,z)}$$

- Equation 30 represents the fundamental solution of the Gaussian beam propagation. It can be seen that both the broadening in r and propagation in z requires lowering the amplitude, such that the energy is conserved,

$$A(r,z) = \frac{1}{w_0{}^2\left(1 + z^2/z_0{}^2\right)} \cdot e^{-\frac{r^2}{w_0{}^2\left(1 + z^2/z_0{}^2\right)}}. \qquad (5.31)$$

- The new beam waist is $w(z) = w_0\sqrt{1 + z^2/z_0{}^2}$.

- The phase dependence reveals more features

$$\phi(r,z) = k_0 z + \frac{k_0 r^2}{2R(z)} - \arg\left(\frac{z}{z_0}\right). \qquad (5.32)$$

- Besides the plane wave phase delay, $k_0 z$, the wavefront radius of curvature R(z) induces additional phase shift, with R(z) defined as

$$R(z) = z\left(1 + \frac{z_0^2}{z^2}\right)$$
(5.33)

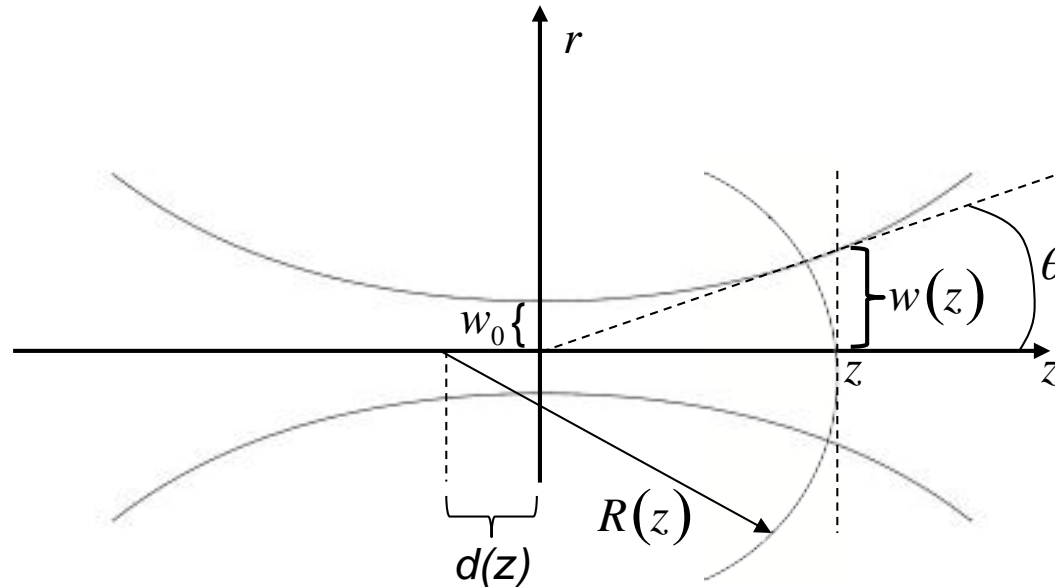- The wavefront is centered a distance $d(z) = 2z_0 / z$ behind the z=0 plane.

Figure 5-3. Propagation of Gaussian beams.

- We can define a far field divergence angle,

$$\theta = \lim_{z \to \infty} \frac{w(z)}{z}$$

$$= \frac{w_0}{z_0} \qquad\qquad (5.34)$$

$$= \frac{\lambda}{\pi w_0}$$

- In far field, $R(z) \simeq z$.

- Perhaps the most surprising result is the phase delay due to the last term in Eq. 33, $\psi(z) = \arg\left(\frac{z}{z_0}\right)$. It can be seen that on propagating from z=0 to ∞, the wavefront is advanced by $\frac{\pi}{2}$. Overall, from $-\infty$ to ∞, there is a total phase shift of π. This peculiar phase shift is referred to as *Gouy's phase*.
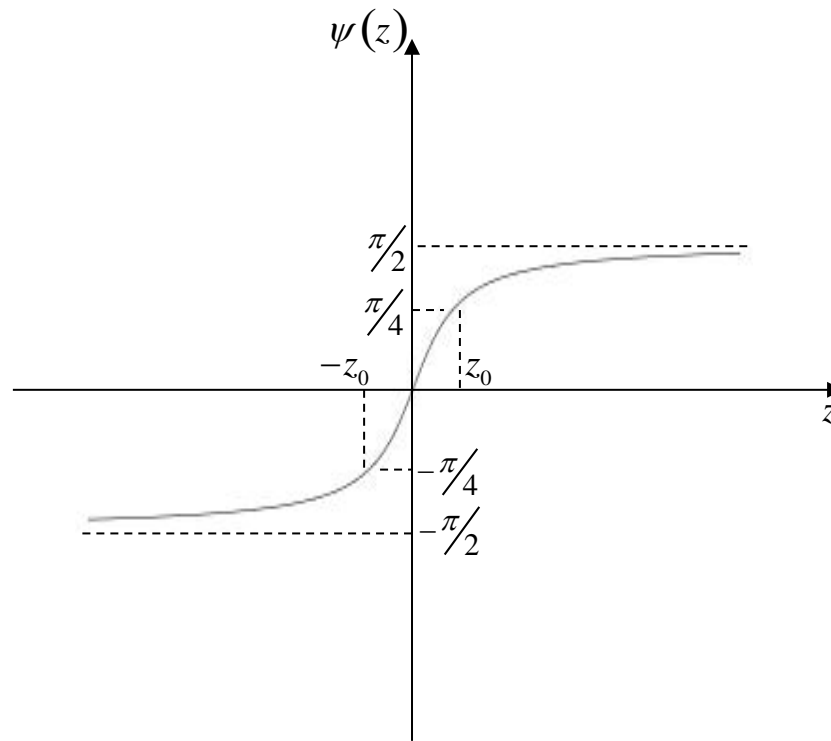
Figure 5-4. Gouy phase.

- This phase advancement contributes to the overall phase velocity v_p along the z axis, such that it exceeds the speed of light in vacuum,

$$v_p = \frac{\omega}{k_{eff}}, \qquad (5.35)$$

- k_{eff} is the effective modulus of the wave vector, defined as

$$k_{eff} = \frac{d}{dz}\left[k_0 z - \psi(z)\right]$$

$$= k_0 - \frac{1/z_0}{1 + \dfrac{z^2}{z_0{}^2}} < k_{0.}$$

(5.36)

- Thus, the contribution of the Gouy phase is always to advance the phase and bring the phase velocity to superluminal values, $v_p > c$.

5.5 Matrix formalism for Gaussian beam propagation.

- We discussed that in the far zone, $z \gg z_0$, we can define a diffraction angle, $\theta = w/z$. In this asymptotic regime, the propagation is well described by straight lines, which recovers geometrical optics.

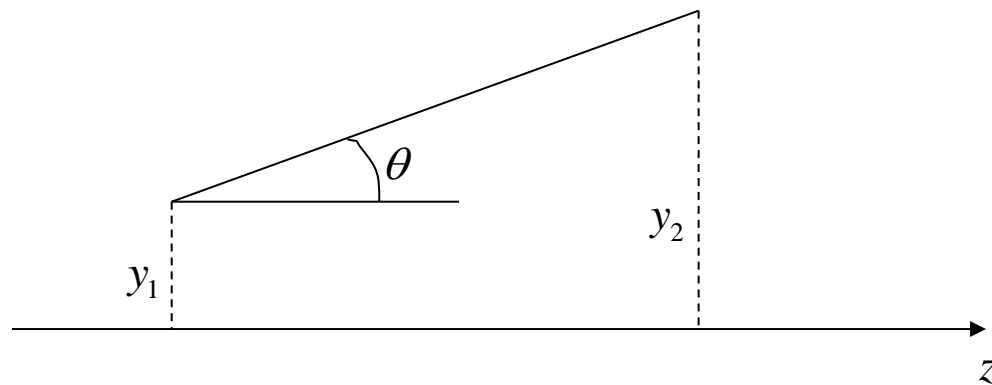- We will develop a *matrix formalism* for retrieving the beam characteristics upon propagation.

Figure 5. Geometrical limit of Gaussian beam propagation.

- Find the transformation rules for the elevation y and angle θ associated with the ray (Fig. 5). Investigate propagation in free space (translation), refraction, and propagation through a thin lens.

5.5.1 Translation.

- In free space propagation, the angle stays constant and the elevation changes,

$$y_2 = y_1 + d\theta_1$$
$$\theta_2 = 0 \cdot y_1 + \theta_1$$

(5.37)

- Equation 37 is valid for very small angles, $\tan\theta \simeq \theta$. This transformation can be expressed in matrix form,

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = M_d \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}, M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

(5.38)

- Equation 38 establishes the characteristic matrix for translation.

5.5.2 Refraction at dielectric interface.

- At the interface between two dielectric media, only the angle changes, according to Snell's law,

$$y_2 = y_1$$

$$\theta_2 = \frac{n_1 \theta_1}{n_2}$$ (5.39)

- Thus, the *refraction matrix* has the form

$$M_R = \begin{pmatrix} 1 & 0 \\ 1 & \dfrac{n_1}{n_2} \end{pmatrix}$$ (5.40)

n_1 $\quad$ n_2
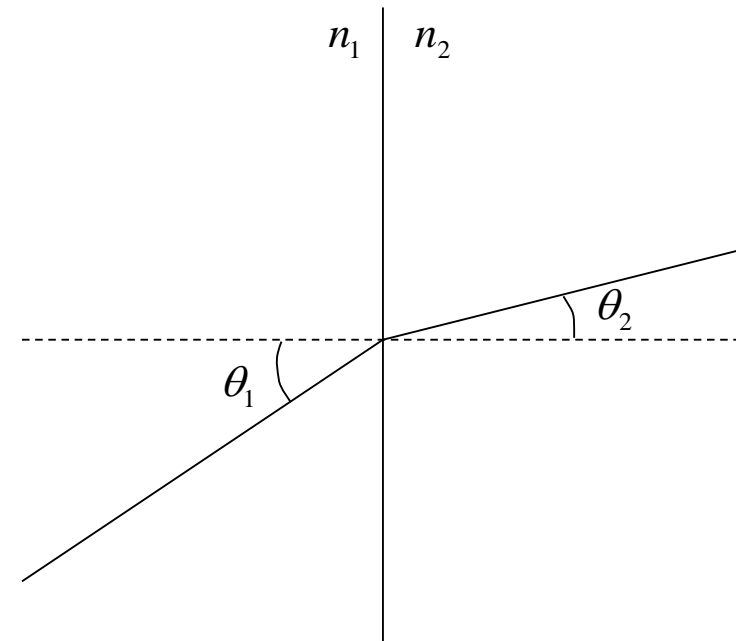
θ_2

θ_1

Figure 5-6. Refraction at a planar interface.

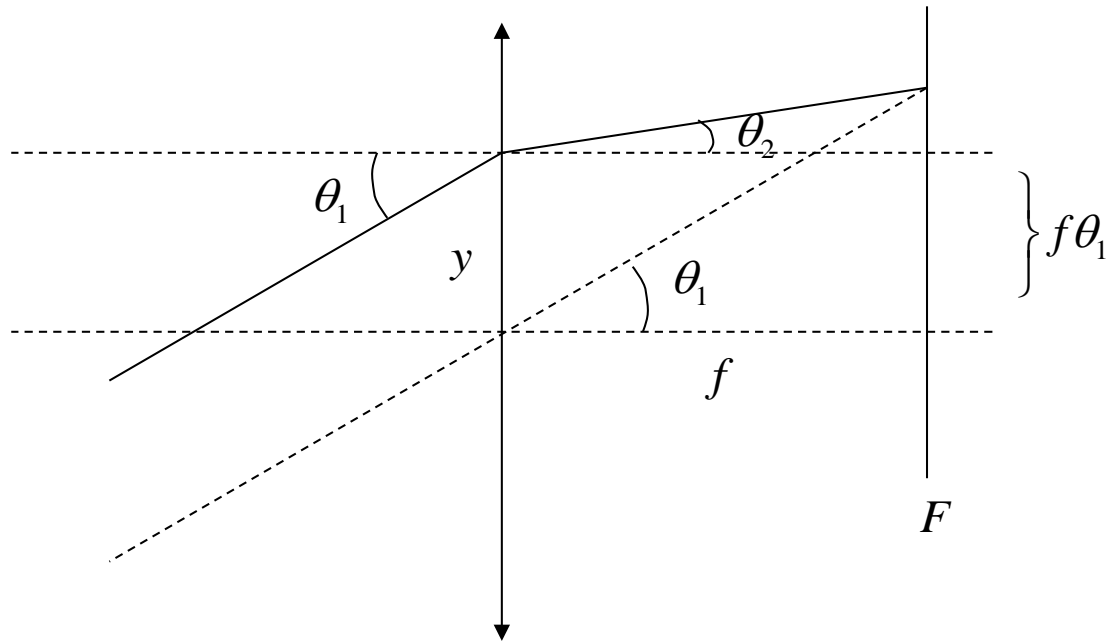## 5.5.3 Propagation through a thin lens.

Figure 5-7. Propagation through a lens.

$$y_2 = y_1$$

$$\theta_2 = \frac{f\theta_1 - y_1}{f} \qquad\qquad (5.41)$$

$$= \theta_1 - \frac{y_1}{f}$$

- Thus the lens transfer matrix is

$$M_f = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix} \qquad\qquad (5.42)$$

- These rules of transforming y and θ are capable of solving complicated geometrical optics problems, combining multiple optical components reduces to matrix multiplication.

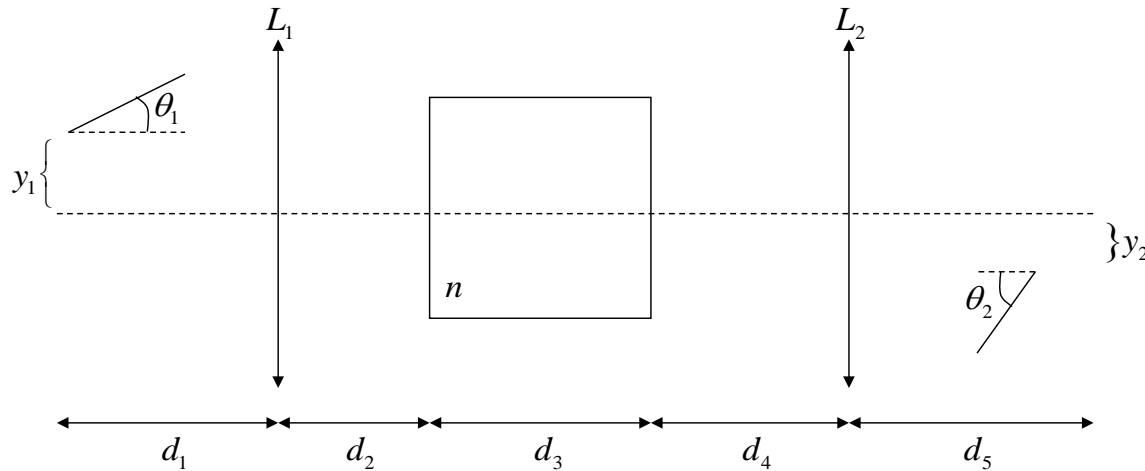- Consider the optical arrangement shown in Fig. 8.

Figure 5-8. Cascading many optical elements.

- The ray starts in the configuration (y_1, θ_1), translates a distance d_1, passes through lens L_1 (focal distance f_1), translates a distance d_2, refracts at the interface with a piece of glass, translates a distance d_3, refracts back to air, translates a distance d_4, passes through lens L_2, and translates a distance L_5, to end in configuration (y_2, θ_2).

- The final configuration can be obtained via matrix multiplication in the proper order,

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = M \begin{pmatrix} y_1 \\ \theta_2 \end{pmatrix} \qquad (5.43)$$

$$M = M_{d_5} \cdot M_{f_2} \cdot M_{d_4} \cdot M_{n1} \cdot M_{d_3} \cdot M_{1n} \cdot M_{d_2} \cdot M_{f_1} \cdot M_{d_1}$$

- The order of matrix multiplication is crucial. To remember the correct order, think of it as "chronological," the first matrix applied to (y_1, θ_1) is the translation matrix M_{d_1}, followed by lens L_1, and so on.

- The transfer matrices associated with optical systems are sometimes referred to as "ABCD matrices," where the matrix elements are specifically labeled as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

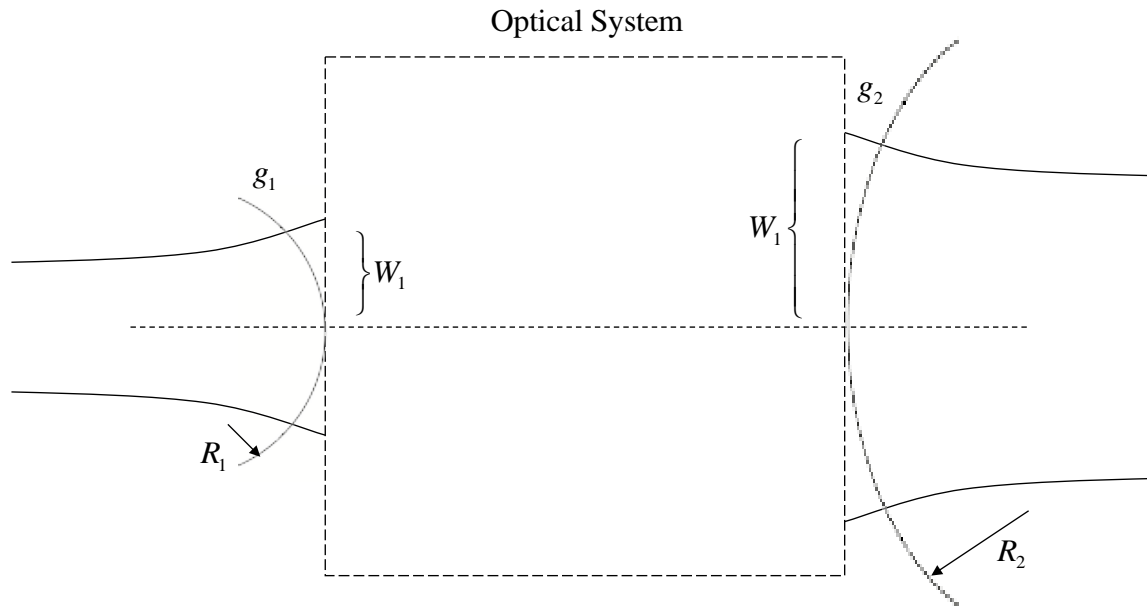- Let us see how this matrix formalism applies to Gaussian beams.

Optical System

Figure 5-9. Propagation of Gaussian beams through an optical system.

- The goal is to find the transformation rules for Gaussian beam parameters, such as the radius of curvature R and waist W.

- Recall that the Gaussian beam is described as (Eq. 26)

$$U(r,z) \propto e^{-\frac{r^2}{c^2(z)}} \qquad (5.44)$$

- We ignore the factor $1\big/c^2(z)$, with $c^2(z)=W^2{}_0\left(1+i\dfrac{z}{z_0}\right)$.

- The z dependence of the exponent $-r^2\big/c^2(z)$ is carried by the complex quantity,

$$g(z)=z-iz_0. \tag{5.45}$$

- $g(z)$ is the *complex beam parameter* and its transformation rules are of interest here. $g(z)$ can be expressed in terms of both R and W, as follows

$$\frac{1}{g(z)}=\frac{1}{R(z)}+i\frac{\lambda}{\pi W^2(z)} \tag{5.46}$$

- A Gaussian beam is fully characterized by the complex beam parameter $g(z)$.

- For an optical system characterized by matrix $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, such as shown in Fig. 9, g_1 and g_2 are related by

$$g_2 = \frac{Ag_1 + B}{Cg_1 + D}. \hspace{3cm} (5.47)$$

- The transfer matrices can be efficiently used to propagate Gaussian beams through complicated optical systems. The overall matrix that characterizes a cascade of optical elements is obtained by simple matrix multiplication.
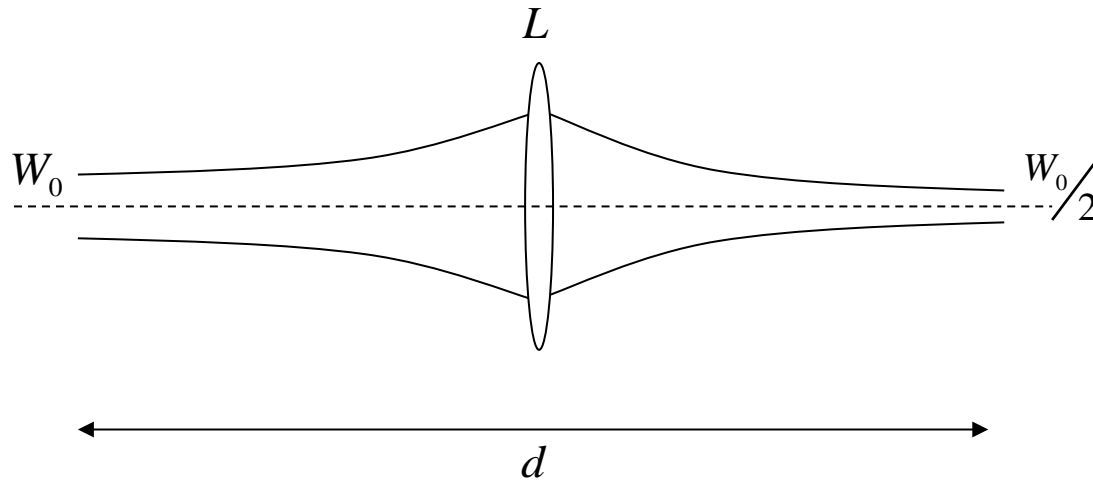
Example 1

Figure 10. Focusing down a Gaussian beam by a convergent lens.

- A Gaussian beam of $\theta_1 = 1\text{mrad}$ divergence beam is focused down by a lens to half its minimum waist at a distance d=1m. What is the focal distance of the lens?

- First, we find that the overall matrix of the system is

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \dfrac{d_1}{f} & d_1 + d_2 - \dfrac{d_1 d_2}{f} \\ -\dfrac{1}{f} & 1 - \dfrac{d_2}{f} \end{pmatrix} \qquad (5.48)$$

- The radius of curvature is infinite at both planes, such that using Eq. 46, we obtain

$$g_1 = j\frac{\pi W_0^{\,2}}{\lambda} = jz_{0_1}$$

$$g_2 = j\frac{\pi W_0^{\,2}}{4\lambda} = jz_{0_2} \qquad (5.49)$$

- Using that $g_2 = \left(Ag_1 + B\right)/\left(Cg_1 + D\right)$, we obtain

$$\frac{z_{0_1} z_{0_2}}{f} = d_1 + d_2 - \frac{d_1 d_2}{f}$$

$$z_{0_2}\left(1 - \frac{d_2}{f}\right) = z_{0_1}\left(1 - \frac{d_1}{f}\right)$$

(5.50)

- Using the information that $d_1 + d_2 = 1\mathrm{m}$, after some manipulations, we obtain $f = 0.19\mathrm{m}$.