# 1. GROUNDWORK

## 1.1. Superposition Principle.

- **This principle states that, for linear systems, the effects of a sum of stimuli equals the sum of the individual stimuli.**
- *Linearity* will be mathematically defined in section 1.2.; for now we will gain a physical intuition for what it means
- *Stimulus* is quite general, it can refer to a force applied to a mass on a spring, a voltage applied to an LRC circuit, or an optical field impinging on a piece of tissue.
- *Effect* can be anything from the displacement of the mass attached to the spring, the transport of charge through a wire, to the optical field scattered by the tissue.
- The stimulus and effect are referred to as *input* and *output* of the *system*. By system we understand the mechanism that transforms the input into output; e.g. the mass-spring ensemble, LRC circuit, or the tissue in the example above.

- The consequence of the superposition principle is that the solution (output) to a complicated input can be obtained by solving a number of simpler problems, the results of which can be summed up in the end.

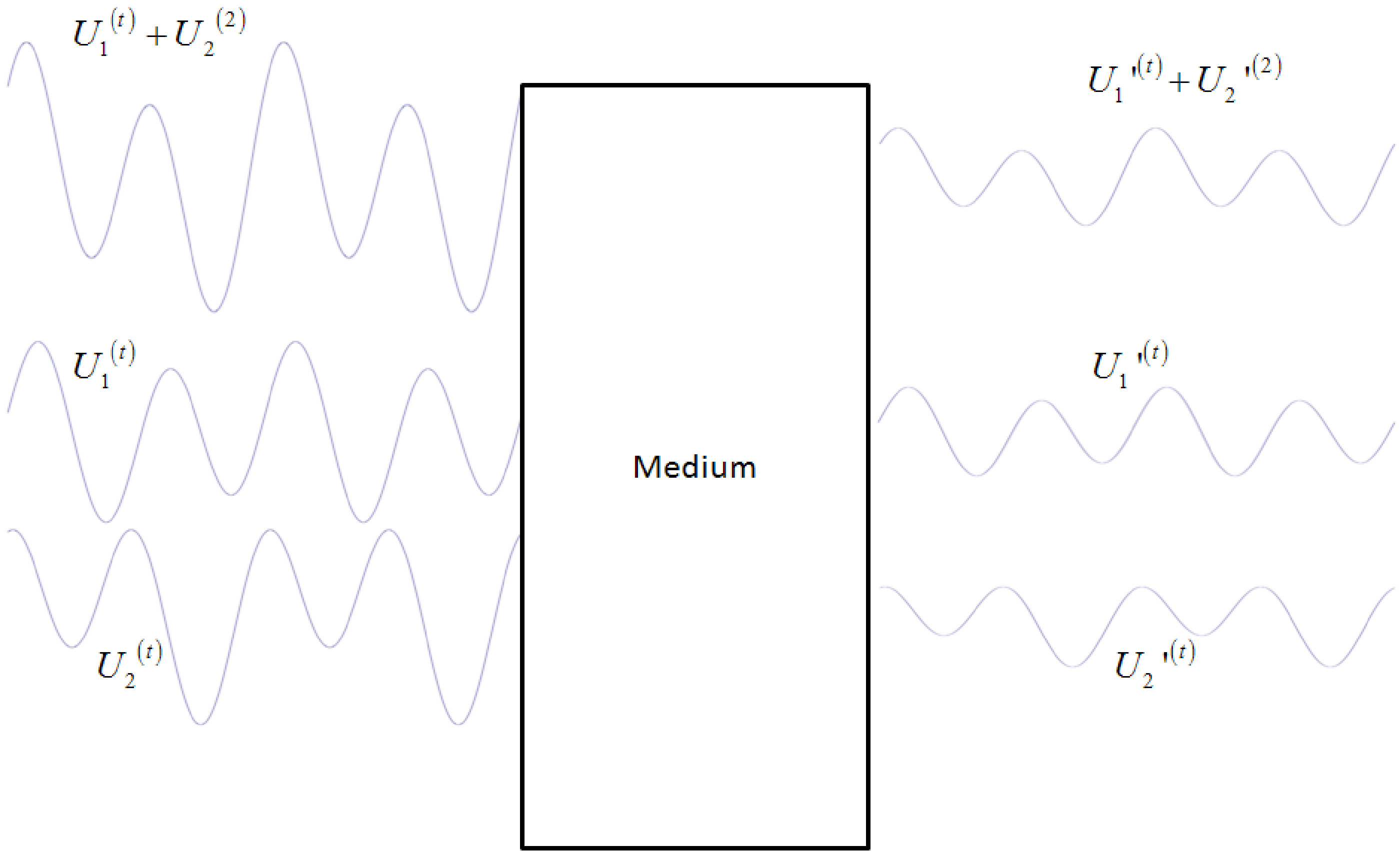



Figure 1. The superposition principle. The response of the system (e.g. a piece of glass) to the sum of two fields is the sum of the output of each field.

- To find the response to the two fields through the system, we have two choices:
  - i) add the two inputs $U_1 + U_2$ and solve for the output;
  - ii) find the individual outputs and add them up, $U'_1 + U'_2$.
- The second option relies on superposition. The superposition principle allows us to decompose $U_1$ and $U_2$ into yet simpler signals, for which the solutions can be easily found.

### *1.1.1. The Green's Function Method.*

- Green's method of solving linear problems refers to "breaking down" the input signal into a succession of pulses that are infinitely thin, expressed by Dirac delta functions. We will deal with temporal responses, spatial responses, or a combination of the two.

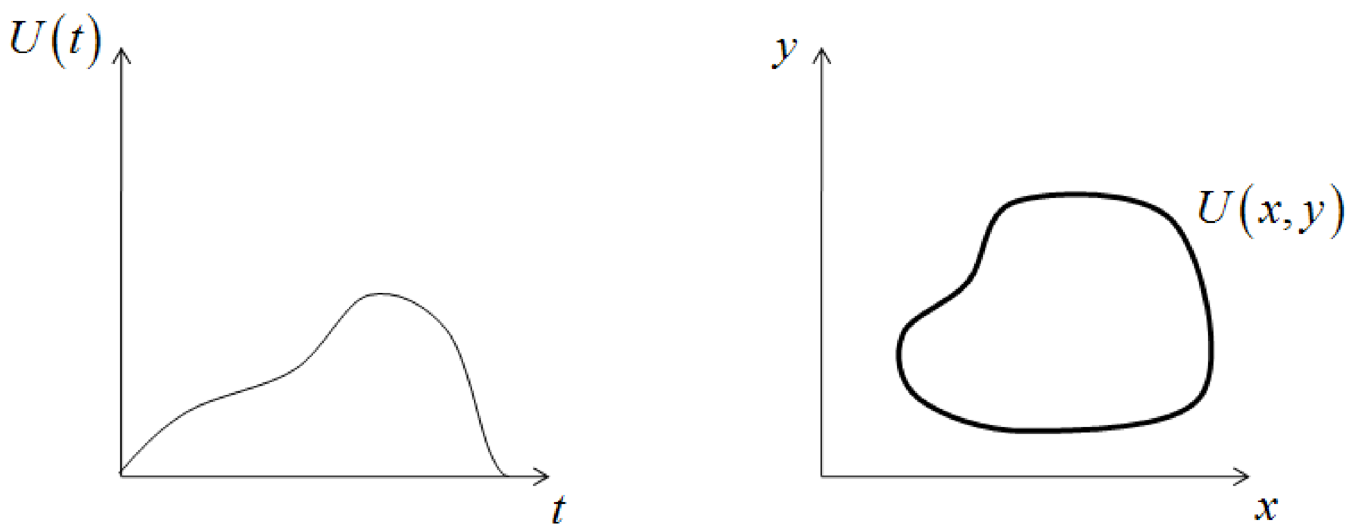



Figure 2. Temporal and spatial field input as distributions of impulses.

- Using a basic property of $\delta$-functions, the input in Fig. 2a can be written as

$$U(t) = \int U(t') \cdot \delta(t - t') dt', \qquad (1.1)$$

- defines $U(t)$ as a summation over infinitely short pulses, each characterized by a position in time, $t - t'$, and strength $U(t')$.

- Exploiting the superposition principle, the response to temporal field distribution can be obtained by finding the response to each impulse and summing the results. This type of problem is useful in dealing with propagation of light pulses through various media.

- The response to the 2D input $U(x, y)$ shown in Fig. 2b can be obtained by solving the problem for each impulse and adding the results.

$$U(x, y) = \iint U(t') \cdot \delta(x - x', y - y') dx' dy' \quad (1.2)$$

- This type of input is encountered in problems related to imaging (works the same way in 1D and 3D as well).

- Green's method is extremely powerful because solving linear problems with impulse input is typically an easy task. The response to such an impulse is called the *Green's function* or the *impulse response* of the system.

## *1.1.2 Fourier Transform Method*

- Another way of decomposing an input is to break it down into sinusoidal signals. Essentially any curve can be reconstructed by summing up such sine waves, as illustrated for both temporal and spatial input signals in Fig. 3.
- Solving a linear problem for a single sinusoid as input is a simple task. The output is the summation of all responses associated with these sinusoids.
- The signals illustrated in Fig. 3 are real, the space and time input are reconstructed from a summation of *cosine* signals. The *Fourier decomposition* of a signal is the generalization of this concept whereby a signal which generally can be complex, is decomposed in terms of $e^{i\omega t}$ (for time) or $e^{i\mathbf{kr}}$ (for space).

and spatial input signals in fig. ...

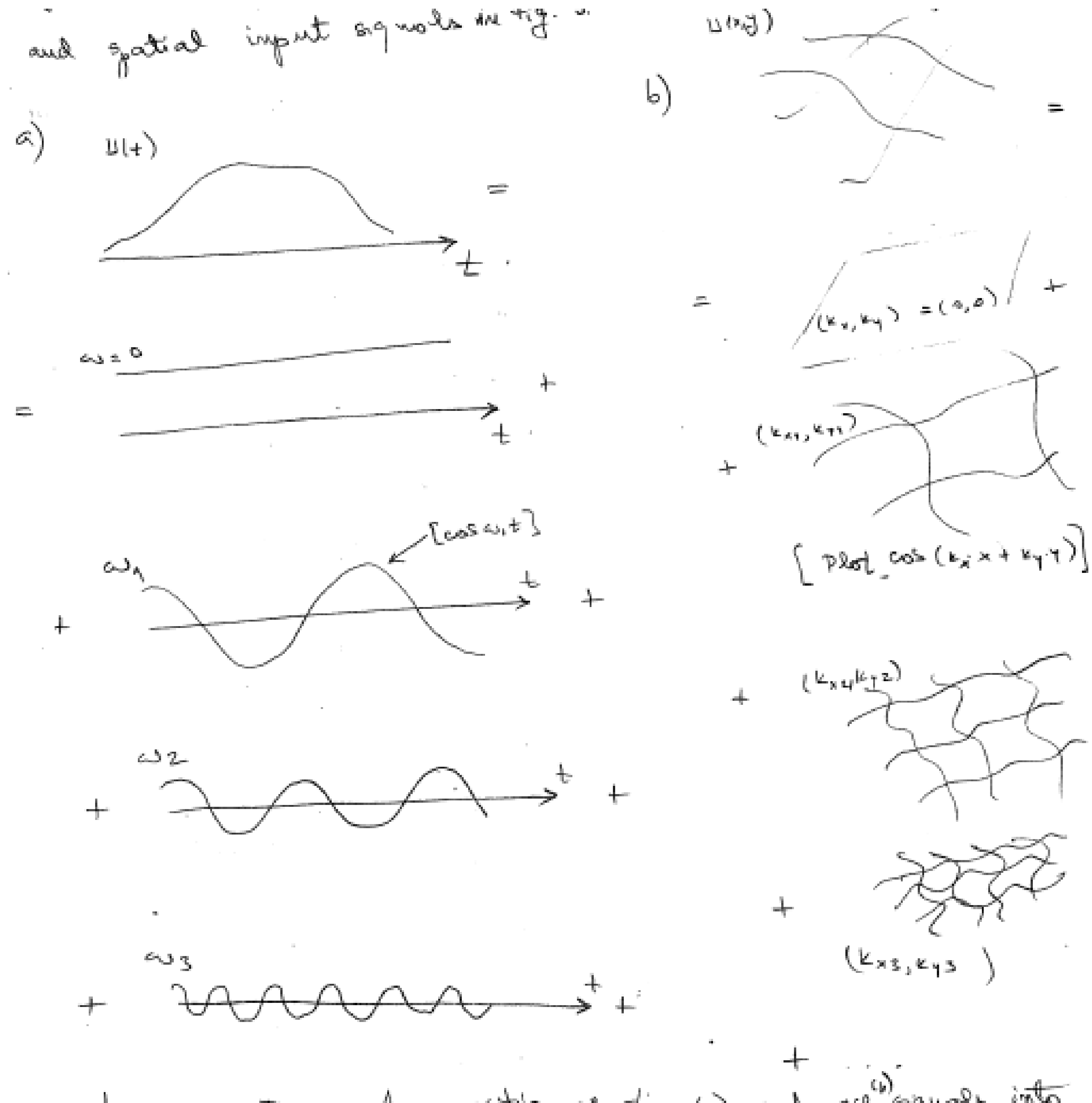



Fig. 3. Decomposition of time (a) and space (b) signals into

## 1.2. Linear Systems.

- Most physical systems can be discussed in terms of the relationship between causes and their effects, i.e. input-output relationships.
- Let us denote $f(t)$ as the input and $g(t)$ as the output of the system (Fig. 4).

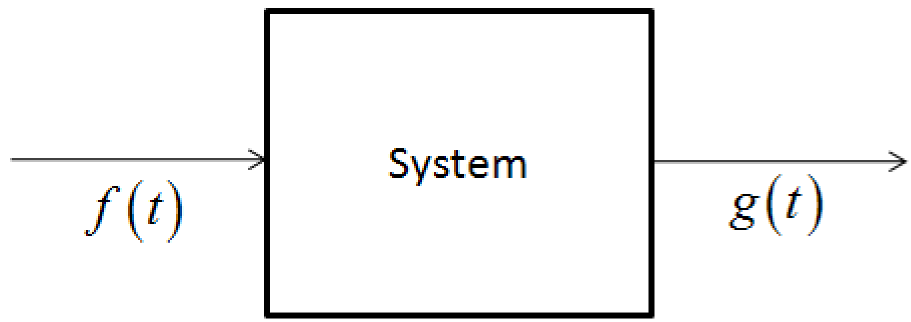



Figure 4. Input-output of a system.

- The system provides a mathematical *transformation*, $L$, which transforms the input into output,

$$L\left[f(t)\right] = g(t) \qquad (1.3)$$

- To fully characterize the system, we must determine the output to all possible inputs, which is virtually impossible. If the system is *linear*, its complete characterization simplifies greatly, as discussed in the following section.

## *1.2.1 Linearity.*

- A system is *linear* if the response to a linear combination of inputs is the linear combination of the individual outputs,

$$\begin{aligned} L\left[a_1 f_1(t) + a_2 f_2(t)\right] &= a_1 L\left[f_1(t)\right] + a_2 L\left[f_2(t)\right] \\ &= a_1 g_1(t) + a_2 g_2(t), \end{aligned} \quad (1.4)$$

- $g_{1,2}$ is the output of $f_{1,2}$, and $a_{1,2}$ are arbitrary constants. The transformation L is referred to as a *linear operator* (an operator is a function that operates on other functions).
- Derive the main property associated with the input-output relationship of a linear system. First, we express an arbitrary input as a sum of impulses

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} f(t'_0) \cdot \delta(t - t') dt' \\ &= \sum_i f(t_i) \cdot \delta(t - t_i)(t_{i+1} - t_i). \end{aligned} \quad (1.5)$$

- In Eq. 5, we expressed the integral as a Riemann summation, which emphasizes the connection with the linearity property. The response to input $f(t)$ is

$$L\left[f(t)\right]=\int_{-\infty}^{\infty} f(t')\cdot L\left[\delta(t-t')\right]dt', \qquad (1.6)$$

- we assumed that the linearity property expressed in Eq. 4 holds for infinite summation.
- Equation 6 indicates that the output to an arbitrary input is the response to an impulse, $L\left[\delta(t-t')\right]$, averaged over the entire domain, using the input (*f*) as the weighting function.
- The system is fully characterized by its *impulse* response, defined as

$$h(t,t')=L\left[\delta(t-t')\right]. \qquad (1.7)$$

- *h* is not a single function, but a family of functions, one for each shift position *t*'.

## *1.2.2. Shift Invariance.*

- An important subclass of systems are characterized by *shift invariance*. For *linear and shift invariant systems*, the response to a shifted impulse is a shifted impulse response,

$$\begin{aligned} L\left[\delta(t-t')\right] &= h(t,t') \\ &= h(t-t'). \end{aligned} \tag{1.8}$$

- The *shape* of the impulse response is independent of the position of the impulse.
- This simplifies the problem and allows us to calculate explicitly the output, $g(t)$, associated with an arbitrary input, $f(t)$. Combining Eqs. 6 and 8, we obtain

$$g(t) = \int_{-\infty}^{\infty} f(t') \cdot h(t-t')\,dt'. \tag{1.9}$$

- Eq. 9 shows that the output for an arbitrary input signal is determined by a single system function , the *impulse response* $h(t)$, or *Green's function*. The integral operation between *f* and *h* is called *convolution*.
- From Eq. 9 we see that if the entire input signal is shifted, say $f(t)$ becomes $f(t-a)$, its output will be shifted by the same amount, $g(t-a)$

$$L\left[f(t-a)\right] = g(t-a) \qquad (1.10)$$

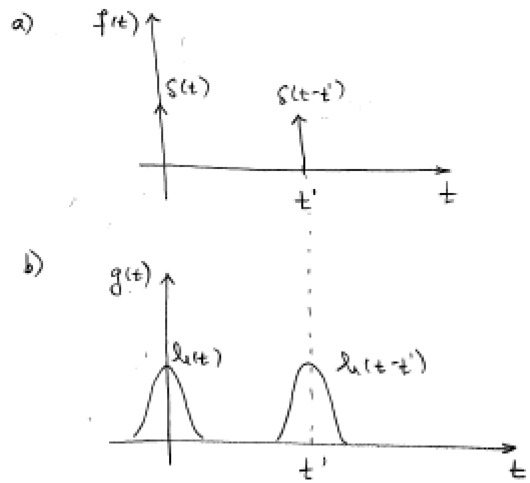



Figure 5. In a linear, shift-invariant system, the response to a pulse shifted by 't'' is the impulse response shifted by t'.

## *1.2.3. Causality.*

- In a *causal system*, the effect cannot precede its cause. The common understanding of causality refers to systems operating on *temporal* signals.
- Let us consider an input signal $f(t)$ and its output $g(t)$.

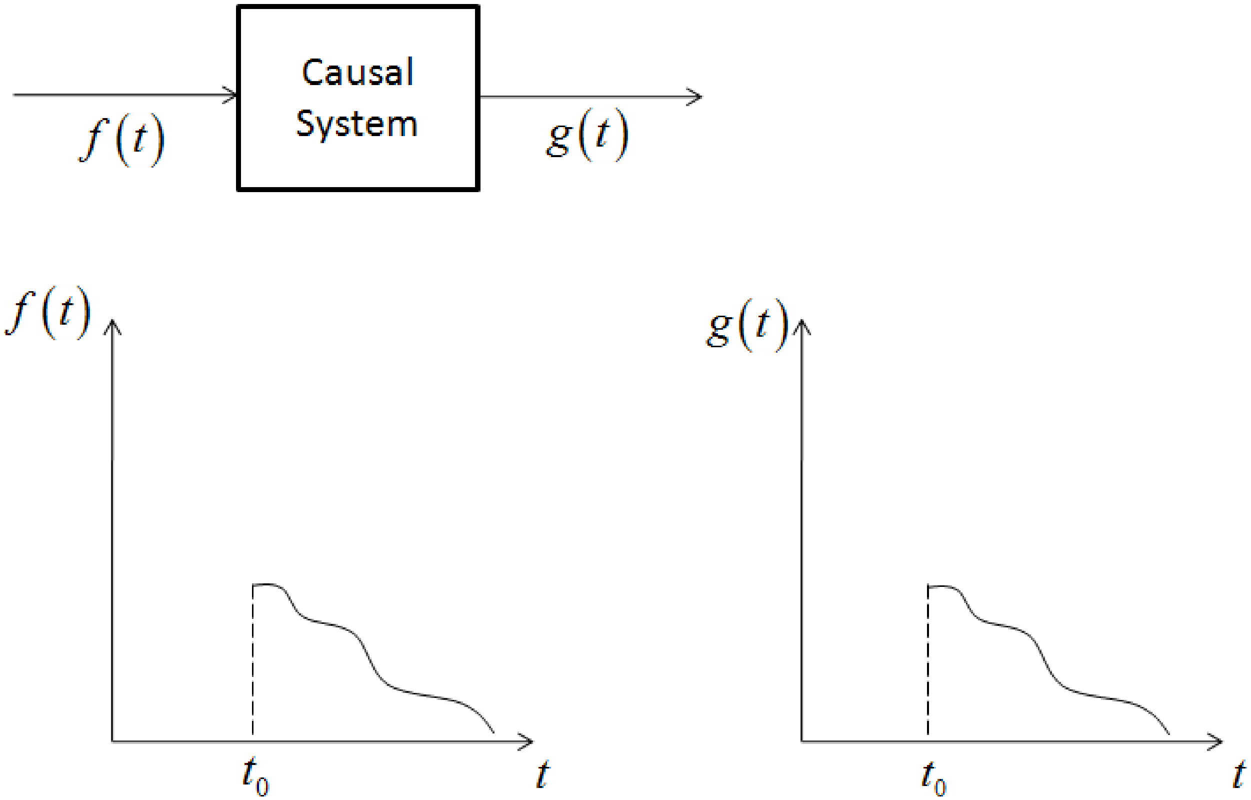



*Figure 7. Input and output for a causal system.*

- Mathematically, causality can be expressed as:

$$\begin{aligned} &\text{if } f(t) = 0 \text{ for } t < t_0 \\ &\text{then } g(t) = 0 \text{ for } t < t_0. \end{aligned} \qquad (1.11)$$

- The output can be written as

$$g(t) = \int_{-\infty}^{\infty} f(t - t')h(t')dt', \qquad (1.12)$$

- where the impulse response, $h(t) = 0$, for $t < t_0$.

- The concept of causality can be extended to other domains. Figure 7 shows an example where the system is the diffraction on an opaque screen. Thus, the screen transforms a spatial distribution of light, $f(x)$, the *input*, into a spatial distribution of diffracted light (the *output*), at a certain distance, $z_0$.

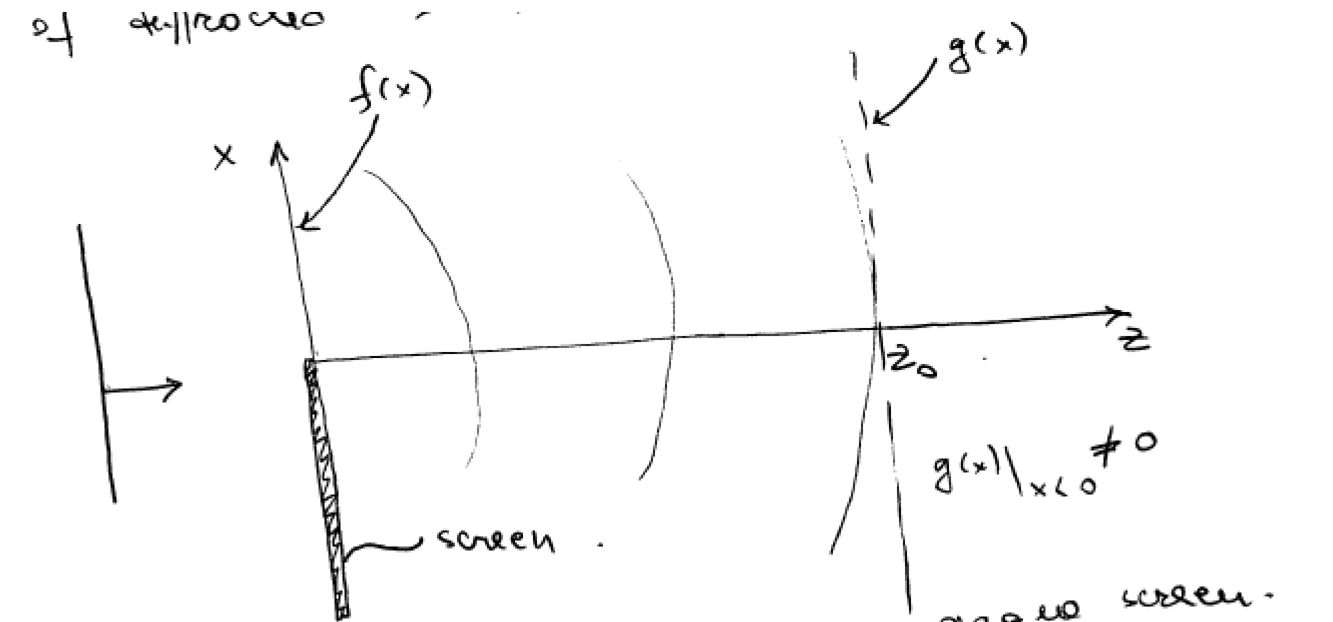



Fig. 7. Diffraction of light on an opaque screen.

- At distance $z_0$ behind the screen, there is a non-zero distribution of field, $g(x)$, below the z-axis, i.e. for $x < 0$. Thus, although the input $f(x) = 0$, for $x < 0$, the output $g(x) \neq 0$, for $x < 0$. We can conclude that diffraction is spatially *non-causal*, so to speak.
- We will discuss in Section 1.4 (Complex analytic signals) a very important property of signals that vanish over a certain semi-infinite domain.

## *1.2.4. Stability.*

- A linear system is *stable* if the response to a bounded input, $f(t)$, is a bounded output, $g(t)$.
- Mathematically, stability can be expressed as

$$\begin{aligned} &\text{if } \left|f(t)\right| < b \\ &\text{then } \left|g(t)\right| < \alpha b, \end{aligned} \qquad (1.13)$$

- $b$ is finite and $\alpha$ is a constant independent of the input.
- The constant $\alpha$ is a characteristic of the system whose meaning can be understood as follows. Let us express the modulus of the output via Eq. 11,

$$\begin{aligned} \left|g(t)\right| &= \left|\int_{-\infty}^{\infty} f(t-t')h(t')dt'\right| \\ &\leq b\int_{-\infty}^{\infty} \left|h(t')\right|dt'. \end{aligned} \qquad (1.14)$$

- Equation 13 proves that if the system is stable, then the impulse response is absolute-integrable. To show that stability is equivalent to $\alpha \cdot \int \left|h(t)\right| dt < \infty$, we need to prove that, conversely, if $a = \infty$, there exists a bounded function that generates an unbounded response. Consider as input

$$f(t) = \frac{h(t)}{\left|h(t)\right|}. \qquad (1.15)$$

- Clearly, $\left|f(t)\right| = 1$, yet its response diverges at the origin,

$$\begin{aligned} g(0) &= \int_{-\infty}^{\infty} f(-t')h(t')dt' \\ &= \int_{-\infty}^{\infty} \frac{h^2(t)}{\left|h(t)\right|} dt = \int_{-\infty}^{\infty} \left|h(t)\right| dt = \infty \end{aligned} \qquad (1.16)$$

- We can conclude that a linear system is stable if and only if its impulse response is modulus-integrable.

## 1.3. The Fourier Transform in Optics

- The Fourier transform and its properties are central to understanding many concepts. Physically, decomposing a signal into sinusoids, or complex exponentials of the form $e^{-i\omega t}$, is motivated by the *superposition principle*, as discussed in Section 1.1.
- For linear systems, obtaining the response for each sinusoidal and summing the responses is always more effective than solving the original problem of an arbitrary input.

## *1.3.1 Monochromatic Plane Waves.*

- There are two types of *complex exponentials*, one describing the *temporal* and the other the *spatial* light propagation.
  - $e^{-i\omega t}$ describes the temporal variation of a *monochromatic* (single frequency) of *angular frequency* $\omega$, $[\omega] = \text{rad/s}$.
  - $e^{ik_x x}$ describes the spatial variation of a plane wave (single direction) propagating along the x-axis, with a *wavenumber* $k_x$, $[k_x] = \text{rad/m}$.
- The two exponents have opposite signs, which is important and can be understood as follows.
- Let us consider a monochromatic plane wave propagating along *x* and also oscillating in time.

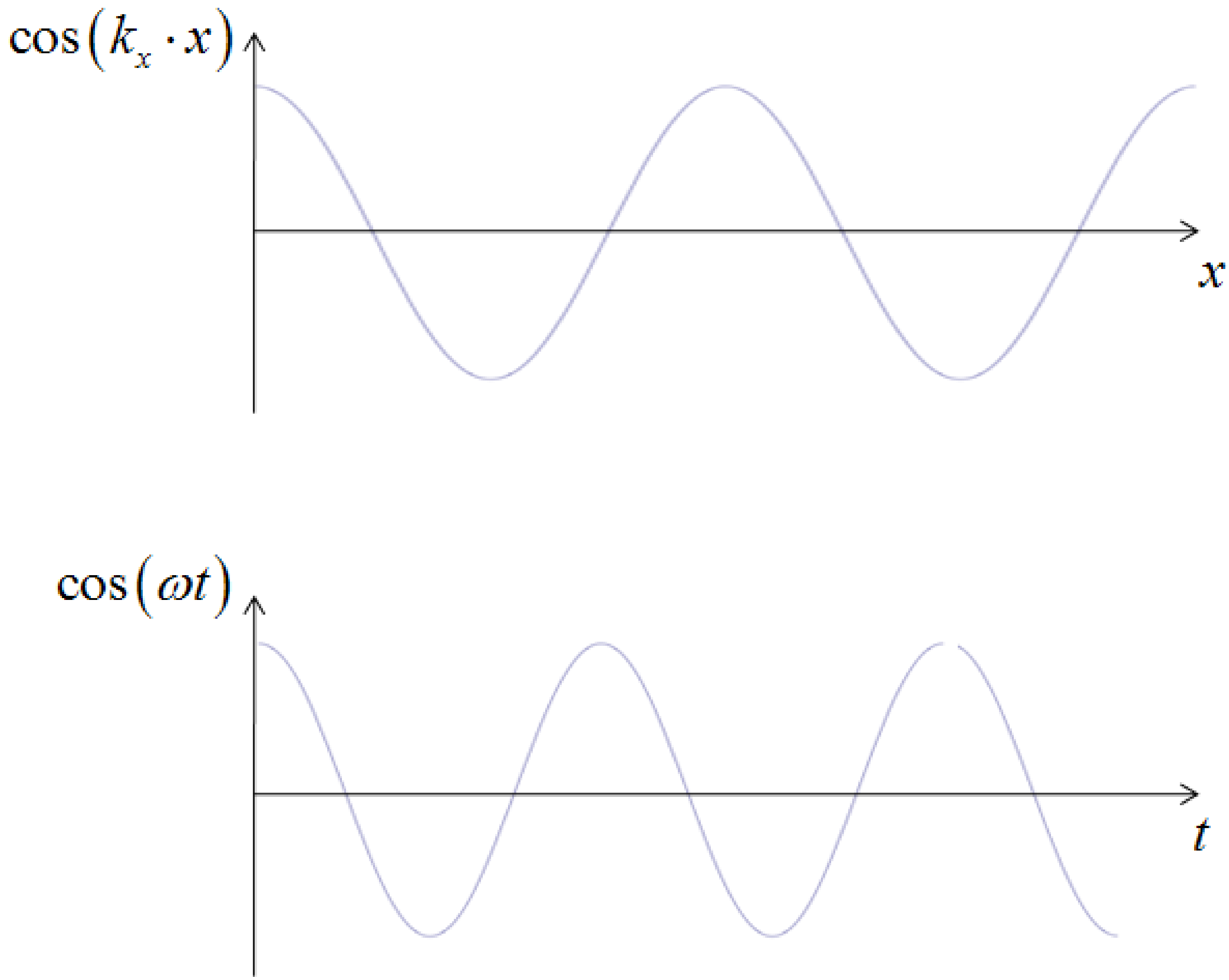



Figure 8. Spatial and temporal variation of (the real part of) a monochromatic plane wave.

- An observer at a fixed spatial position $x_0$, "sees" the wave pass by with a *temporal phase*, $\phi(t) = -\omega t$. Another observer has the ability to freeze the wave in time $t = t_0$ and "walk" by it along the positive x-axis; the *spatial phase* will have the opposite sign, $\phi(x) = k_x x$.
- An analogy that further illustrates this sign change is to consider a travelling train whose cars are counted by the two observers above. The first observer, from a fixed position $x_0$, sees the train passing by with its locomotive, then car 1, car 2, etc. The second observer walks by the train that is now stationary, and sees the cars in reverse order towards the locomotive.
- We will use the *complex exponential* $e^{-i\omega t + \mathbf{k}\cdot\mathbf{r}}$ to denote a monochromatic plane wave (the dot product $\mathbf{k}\cdot\mathbf{r}$ appears whenever the direction of propagation is not parallel to an axis of the coordinate system.

- An equivalent function is $e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}$. Both functions are valid because physically we only can define phase differences and not absolute phases; then the sign of a phase shift is arbitrary. However the opposite sign relationship between the temporal and spatial phase shift must be reinforced, precisely because it is a *relative* relationship.

## *1.3.2. $e^{-i\omega t+i\mathbf{k}\cdot\mathbf{r}}$ as Eigenfunction.*

- A fundamental property of linear systems is that the response to a complex exponential is also a complex exponential,

$$L\left(e^{-i\omega t+\mathbf{kr}}\right)=\alpha\cdot e^{-i\omega t+i\mathbf{k}\cdot\mathbf{r}}, \qquad (1.17)$$

- $\alpha$ is a constant.
- A function that maintains its shape upon transformation by the system operator $L$ is called an *eigenfunction*, or *normal function* of the system. An eigenfunction is not affected by the system except for a multiplicative (scaling) constant.
- Let us prove Eq. 16 for a system that only operates in time domain for simplicity. Let $g(t)$ be the response of the system to $e^{-i\omega t}$,

$$L\left(e^{-i\omega t}\right)=g(t) \qquad (1.18)$$

- If we invoke the *shift invariance* property, $L\left[f\left(t-t'\right)\right]=g\left(t-t'\right)$, and apply to the input $e^{-i\omega t}$, we obtain

$$L\left[e^{-i\omega\left(t-t'\right)}\right]=g\left(t-t'\right) \qquad (1.19)$$

- $e^{-i\omega\left(t-t'\right)}=e^{i\omega t'}\cdot e^{-i\omega t}$, which for a fixed *t*' is the original $e^{-i\omega t}$ multiplied by a constant. Applying the linearity property and combining with Eq. 19, we obtain

$$\begin{aligned} L\left[e^{i\omega t'}\cdot e^{-i\omega t}\right] &= e^{i\omega t'}g\left(t\right) \\ &= g\left(t-t'\right) \end{aligned} \qquad (1.20)$$

- Equation 20 holds for any *t*, thus, for t=0, Eq. 20 becomes

$$g\left(-t'\right)=g\left(0\right)\cdot e^{i\omega t'}. \qquad (1.21)$$

- Note that *t*' is arbitrary in Eq. 21. If we denote the constant $g\left(0\right)$ by $\alpha$, we finally obtain ($-t'=t$),

$$L\left(e^{-i\omega t}\right)=\alpha\cdot e^{-i\omega t}, \qquad (1.22)$$

- This proves the temporal part of Eq. 16. Of course, the same proof applies to the spatial signal $e^{i\mathbf{kr}}$.
- The fact that $e^{-i\omega t+i\mathbf{kr}}$ is an eigenfunction implies that a signal does not change frequency upon transformation by the linear system, **i.e. in linear systems, the frequencies do not "mix."** This is why linear problems are solved most efficiently in the frequency domain.
- In the following section we discuss the Fourier transform, which is the mathematical transformation that allows spatial and temporal signals to be expressed in their respective frequency domains.

## *1.3.3. The 1D Fourier Transform.*

- Consider a function of real variable *t* and with generally complex values, we define the integral

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{i\omega t} dt. \qquad (1.23)$$

- If the integral exists for every $\omega \in \mathbb{R}$, the function $\tilde{f}(\omega)$ defines the Fourier transform of $f(t)$.

- If we multiply both sides of Eq. 22 by $e^{-i\omega t'}$ and integrate over $\omega$, we obtain what is called the *inversion formula*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) \cdot e^{-i\omega t} d\omega. \qquad (1.24)$$

- where we used the property of the $\delta$-function,

$$\int_{-\infty}^{\infty} e^{-i\omega(t-t')} dt = \delta(t-t') \qquad (1.25)$$

- Equation 24 states that function *f* can be written as a superposition of complex exponentials, $e^{-i\omega t}$, with the Fourier transform, $\tilde{f}$ (generally complex), assigning the proper amplitude and phase to each exponential.
- For function *f* to have a Fourier transform, i.e. the integral in Eq. 23 to exist, the following conditions must be met:
    - a) *f* must be modulus-integrable,

$$\int_{-\infty}^{\infty} \left| f\left(t\right) \right| dt < \infty. \qquad (1.26)$$

    - b) *f* has a finite number of discontinuities within any finite domain,
    - c) *f* has no infinite discontinuities.
- Many functions of interest in optics do not satisfy condition a) and thus strictly speaking do not have a Fourier transform. Examples include the sinusoidal function, step function, and $\delta$-function.

- This type of function suffers from singularities, and can be further described by defining *generalized Fourier transforms* via $\delta$-functions
- Any stationary signal (whose statistical properties, such as its average, do not change in time) violates condition *a*.
- Spectral properties of *stationary random processes* have been studied in depth by Wiener [Wiener 1930], who developed a new theory called the *generalized harmonic analysis* to describe them.
- **Wiener showed that the *power spectrum* is well defined for signals that do not have a Fourier transform, as long as their autocorrelation function is well defined. Formally, optical fields should be described statistically, using the theory for random processes.**

- It is common in practice to use Fourier transform to describe optical field distributions in both time and space. This apparent contradiction can be understood as follows.
- In real situations of practical interest, we always deal with fields that are of finite support both temporally and spatially. For any signal $f(t)$ that violates Eq. 26, we can define a truncated version, $\underline{f}$, described as

$$\underline{f}(t) = \begin{cases} f(t), \text{ if } t \in \left(-\tau/2, \tau/2\right) \\ 0, \text{ rest} \end{cases} \qquad (1.27)$$

- For most functions of interest, their *truncated versions* now do satisfy Eq. 26,

$$\int_{-\infty}^{\infty} \left|\underline{f}(t)\right| dt = \int_{-\tau/2}^{\tau/2} \left|f(t)\right| dt < \infty. \qquad (1.28)$$

- We now can use the Fourier transform of $\underline{f}$ rather than *f*.

- Let us consider $f(t)=\cos(\alpha t)$. Thus, the Fourier transform of the respective truncated function is

$$\underline{\tilde{f}}(\omega)=\int_{-\tau/2}^{\tau/2}\cos(\alpha t)\cdot e^{i\omega t}dt$$

$$=\frac{1}{2}\int_{-\tau/2}^{\tau/2}\left(e^{i\alpha t}+e^{-i\alpha t}\right)\cdot e^{i\omega t}dt \qquad (1.29)$$

$$=\frac{\sin\left[(\omega+\alpha)\tau/2\right]}{\omega+\alpha}+\frac{\sin\left[(\omega-\alpha)\tau/2\right]}{\omega-\alpha}.$$

- We can now take the limit $\tau\to\infty$ and, invoking the properties of the $\delta$-function, we obtain

$$\tilde{f}(\omega)=\lim_{\tau\to\infty}\underline{\tilde{f}}(\omega)$$
$$=\delta(\omega+\alpha)+\delta(\omega-\alpha) \qquad (1.30)$$

- This example demonstrates how the concept of Fourier transforms can be generalized using the singular function $\delta$.
- In the following section we discuss some important theorems associated with the Fourier transform, which are essential in solving linear problems.

## *1.3.4. Properties of the Fourier Transform.*

- We have seen how the superposition principle allows us to decompose a general problem into a number of simpler problems. The Fourier transform is such a decomposition, which is commonly used to solve linear problems.
- Let us review a set of properties (or theorems) associated with the Fourier transform, which are of great practical use. We present optics examples both for time and 1D spatial domain, where those mathematical theorems apply
- We will use the symbol $\leftrightarrow$ to indicate a Fourier relationship, e.g. $f(t) \leftrightarrow \tilde{f}(\omega)$.
- Occasionally, the operator $F$ will be used to denote the same, e.g. $F\left[f(t)\right] = \tilde{f}(\omega)$.

## *a) Linearity*

- The Fourier transform operator is linear, i.e.

$$F\left[a_1 f_1(t)+a_2 f_2(t)\right]=a_1\tilde{f}_1(\omega)+a_2\tilde{f}_2(\omega).(1.31)$$

- Equation 31 is easily proven by using the definition of the Fourier transform.

## *b) Shift property*

- Describes the frequency domain effect of shifting a function by a constant

$$f\left(t-t_0\right)\leftrightarrow \tilde{f}\left(\omega\right)e^{i\omega t_0} \qquad (1.32)$$

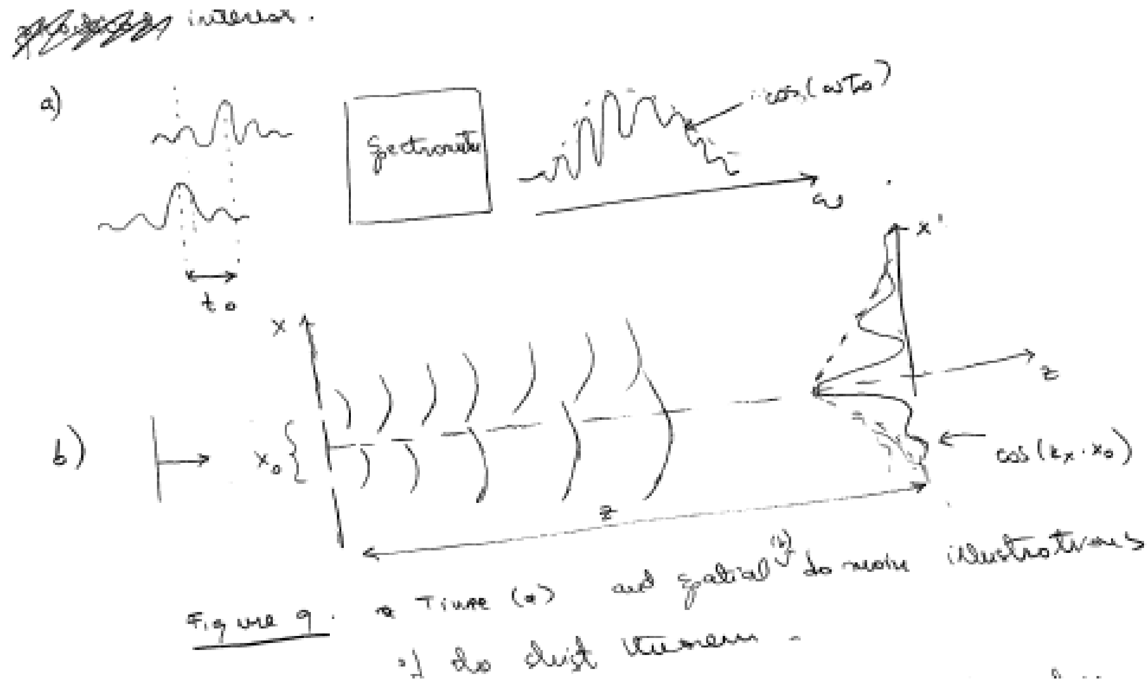
- This property applies equally well to a frequency shift

$$\tilde{f}\left(\omega-\omega_0\right)\leftrightarrow f\left(t\right)\cdot e^{-i\omega t_0} \qquad (1.33)$$

- Equations 32 and 33 have their analogs for the 1D spatial domain

$$\begin{aligned} f\left(x-x_0\right)&\leftrightarrow \tilde{f}\left(k_x\right)\cdot e^{-ik_x\cdot x_0} \\ \tilde{f}\left(k_x-k_0\right)&\leftrightarrow f\left(x\right)\cdot e^{ik_x\cdot x_0} \end{aligned} \qquad (1.34)$$

- Note the sign difference between Eqs. 32 and 34a, as well as 33 and 34b. This is due to the monochromatic plane wave being described by $e^{-i\omega t+k_x\cdot x}$.
- For a pair of identical pulses shifted in time by $t_0$, $u\left(t\right)$, $u\left(t-t_0\right)$, the spectrum measured by the spectrometer is modulated by $\cos\left(\omega t_0\right)$. The spectrometer detects the power spectrum, i.e. $\left|\tilde{u}\left(\omega\right)+\tilde{u}\left(\omega\right)e^{i\omega t_0}\right|^2 \propto \cos\left(\omega t_0\right)$.




- **To illustrate the same property in the spatial domain, we anticipate that upon propagation in free space, a given input field, $u(x)$, is Fourier transformed to $u(k_x)$, with $k_x = \frac{2\pi x'}{\lambda z}$, $\lambda$ the wavelength.** We use this spatial Fourier transform property to describe the full analog to the time domain problem.

- Illuminating two apertures shifted by $x_0$, generated in the far field an intensity distribution modulated by $\cos\left(k_x x_0\right)$, i.e. $\left|\tilde{u}\left(k_x\right)+\tilde{u}\left(k_x\right)e^{ik_x \cdot x_0}\right|^2 \propto \cos\left(k_x \cdot x_0\right)$.
- Here, $\tilde{u}\left(k_x\right)$ is the Fourier transform of one pulse.

### *c) Parseval's theorem*

- This theorem, sometimes referred also by Rayleigh's theorem, states the energy conservation,

$$\int_{-\infty}^{\infty} \left| f\left(t\right) \right|^2 dt = \int_{-\infty}^{\infty} \left| \tilde{f}\left(\omega\right) \right|^2 d\omega$$

$$\int_{-\infty}^{\infty} \left| f\left(x\right) \right|^2 dx = \int_{-\infty}^{\infty} \left| \tilde{f}\left(k_x\right) \right|^2 dk_x \qquad (1.35)$$

- Equations 35a and b show that the total energy of the signal is the same, whether it is measured in time (space) or frequency domain.

## *d) Similarity theorem*

- This theorem establishes the effect that scaling one domain has on the Fourier domain,

$$f(at) \to \frac{1}{|a|} \cdot \tilde{f}\left(\frac{\omega}{a}\right)$$
$$f(ax) \to \frac{1}{|a|} \cdot \tilde{f}\left(\frac{k_x}{a}\right) \qquad (1.36)$$

- The similarity theorem provides an inductive relationship between a function and its Fourier transform, namely, **the narrower the function the broader its Fourier transform and vice-versa.**

Figure 10 gives a temporal and a spatial illustration of Eqs. 31a-b.

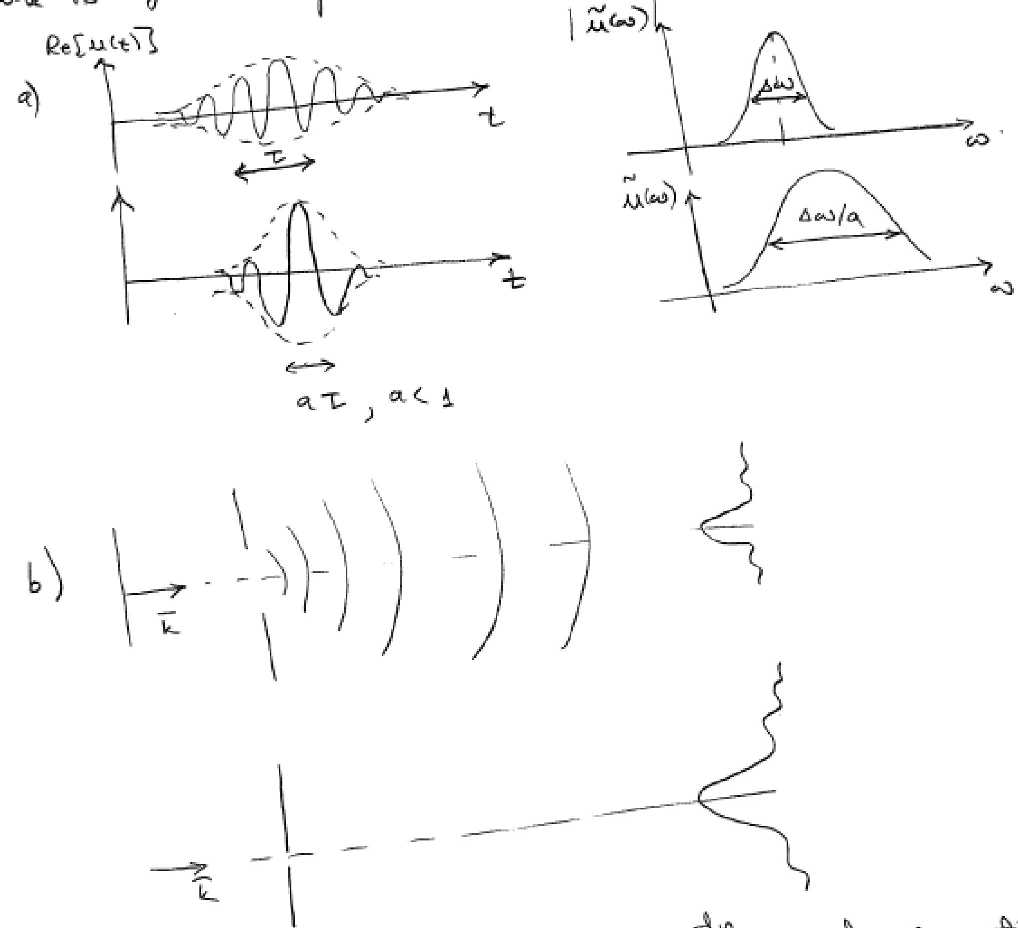



Fig. 10. The similarity theorem in the 1D temporal and spatial domain
a) Shortening light pulse by factor corresponds to broadening
[illegible] same factor. b) The

## *e) Convolution theorem*

- This theorem provides an avenue for calculating integrals that describe the response of linear shift invariant systems. Generally, the convolution operation (on integral) of two functions *f* and *g* is defined as

$$f *_t g = \int_{-\infty}^{\infty} f(t') \cdot g(t - t') dt'$$

$$f *_x g = \int_{-\infty}^{\infty} f(x') \cdot g(x - x') dx' \qquad (1.37)$$

- In the convolution operation function, *g* is flipped, shifted, and multiplied by *f*. The area under this product represents the convolution evaluated at the particular shift value. To evaluate the convolution over a certain domain, the procedure is repeated for each value of the shift.

- Note that $f * g$ operates in the same space as *f* and *g*. The *convolution theorem* states that in the frequency domain the convolution operation becomes a product,

$$\begin{aligned} f *_t g &\to \tilde{f}(\omega)\cdot\tilde{g}(\omega) \\ \tilde{f} *_\omega \tilde{g} &\to f(t)\cdot g(t) \\ f *_x g &\to \tilde{f}(k_x)\cdot\tilde{g}(k_x) \\ \tilde{f} *_{k_x} \tilde{g} &\to f(x)\cdot g(x) \end{aligned} \qquad (1.38)$$

- Equations 33a-b reiterate that linear problems should be solved in the frequency domain, where the output of system can be calculated via simple multiplication.
- **We found in Section 1.2.1 that the output *g* of a linear system is the convolution between the input, *f*, the impulse response, *h*,**

$$g(t) = f * h \qquad \textbf{(1.39)}$$

- **Thus, the convolution theorem allows us to calculate the frequency domain response via a simple multiplication,**

$$\tilde{g}(\omega) = \tilde{f}(\omega) \cdot \tilde{h}(\omega) \qquad \textbf{(1.40)}$$

- There are several other useful properties associated with the convolution operation, which can be easily proven

$$
\begin{aligned}
& f * g = F^{-1}\left[ \tilde{f}(\omega) \cdot \tilde{g}(\omega) \right] \\
& f * g = g * f \\
& f * (g * h) = (f * g) * h \\
& f * (g + h) = f * g + f * h \\
& f * g * h \rightarrow \tilde{f} \cdot \tilde{g} \cdot \tilde{h} \\
& f * (g \cdot h) \rightarrow \tilde{f} \cdot \left( \tilde{g} * \tilde{h} \right)
\end{aligned}
\tag{1.41}
$$

## *f) Correlation Theorem*

- The correlation operation differs slightly from the convolution, in the sense that, under the integral, the argument of *g* has the opposite sign,

$$f \otimes_t g = \int_{-\infty}^{\infty} f(t') \cdot g(t' - t)\, dt'$$
$$f \otimes_x g = \int_{-\infty}^{\infty} f(x') \cdot g(x' - x)\, dx' \qquad (1.42)$$

- In the frequency domain, the correlation function also becomes a product, between one Fourier transform and the conjugate of the other Fourier transform

$$\begin{aligned} f \otimes_t g &\to \tilde{f}(\omega) \cdot \tilde{g}^*(\omega) \\ \tilde{f} \otimes_\omega \tilde{g} &\to f(t) \cdot g^*(t) \\ f \otimes_x g &\to \tilde{f}(k_x) \cdot \tilde{g}^*(k_x) \\ \tilde{f} \otimes_{k_x} \tilde{g} &\to f(x) \cdot g^*(x) \end{aligned} \quad (1.43)$$

- Note that if $g$ is even Eqs. 43a and 43c are the same as 38a and c (the same is true for the other pairs of equations if $\tilde{g}$ is even).

# 2. THE 2D FOURIER TRANSFORM

## 2.1. Definition

- So far, we discussed 1D Fourier transformations. In studying imaging, the concept must be generalized to 2D and 3D functions. Diffraction and 2D image formation are treated efficiently via 2D Fourier transforms, while light scattering and tomographic reconstructions require 3D Fourier transforms.
- A 2D function *f* can be reconstructed from its Fourier transform as

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(k_x,k_y)\cdot e^{i(k_x\cdot x+k_y\cdot y)}dk_x dk_y. \quad (1.44)$$

- The inverse relationship reads

$$\tilde{f}(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\cdot e^{-i(k_x\cdot x+k_y\cdot y)}dxdy. \quad (1.45)$$

- Thus $\tilde{f}(k_x, k_y)$, a complex function, sets the amplitude and phase associated with the sinusoidal of frequency $\mathbf{k} = (k_x, k_y)$. The contours of constant phase are

$$\begin{aligned}\phi(x,y) &= k_x \cdot x + k_y \cdot y \\ &= \text{const.}\end{aligned} \tag{1.46}$$

- Equation 3 can be expressed as

$$\begin{aligned}\phi(x,y) &= |k|\left( x \cdot \frac{k_x}{|k|} + y \cdot \frac{k_y}{|k|} \right) \\ &= \text{const,}\end{aligned} \tag{1.47}$$

- $|k| = \sqrt{k_x^{\ 2} + k_y^{\ 2}}$ .

- From Eq. 4, the direction of the contour makes an angle $\frac{k_x}{|k_x|}$ with the x-axis and has a wavelength $\Lambda = 2\pi / |k|$ (Fig. 1).

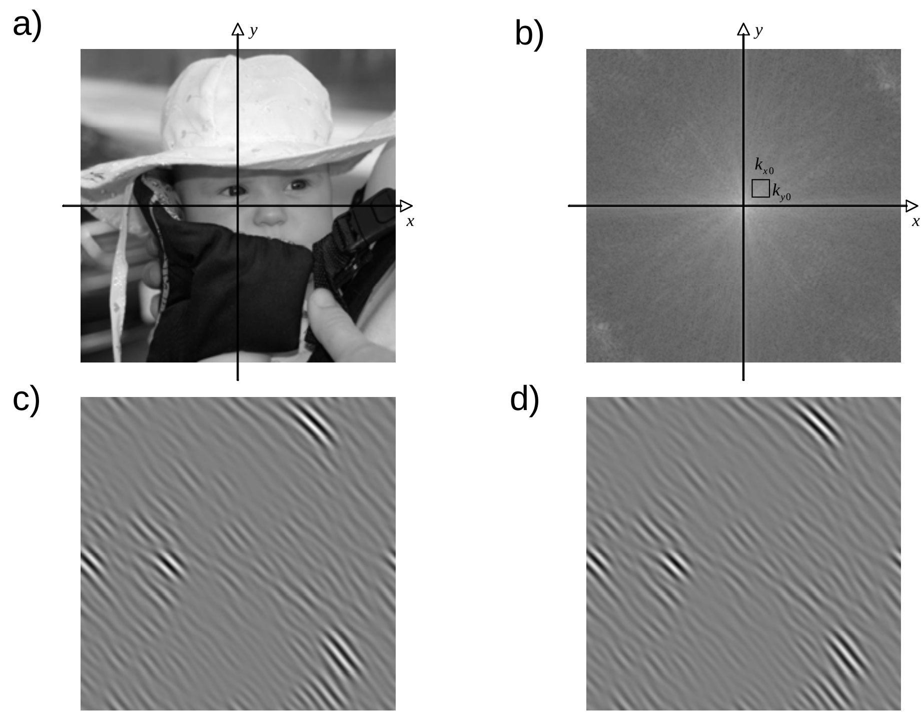



*Figure 1. a) Example of a 2D function f(x,y). b) The modulus of the Fourier transform (i.e. spectrum). c) The (real) Fourier component associated with the frequency ($k_{x0}$,$k_{y0}$) indicated by the square region in b. d) The (imaginary) Fourier component associated with the frequency ($k_{x0}$,$k_{y0}$) indicated by the square region in b.*

- Eq. 1 indicates that 2D function *f* (e.g. an image) is a superposition of waves of the type shown in Fig. 1c, with appropriate amplitude and phase for each frequency $(k_x, k_y)$. The Fourier transform $\tilde{f}(k_x, k_y)$ assigns these amplitudes and phases for each frequency component $(k_x, k_y)$.

## 2.2. Two-dimensional convolution

- The *convolution* operation between two 2D functions $f(x, y)$ and $g(x, y)$ is

$$f *_{xy} g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')g(x - x', y - y')dx'dy' \quad (1.48)$$

- $g$ is rotated by 180° about the origin due to the change of sign in both x' and y', then displaced, and the products integrated over the plane.
- One can encounter one-dimensional convolutions of 2D functions,

$$f *_{x} g = \int_{-\infty}^{\infty} f(x', y')g(x - x', y')dx' \quad (1.49)$$

- One example of such convolutions may occur when using cylindrical optics.

- The 2D *cross-correlation* integral of $f$ and $g$ is

$$f \otimes_{xy} g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y') \cdot g(x+x', y+y')dx'dy'.(1.50)$$

- Like in the 1D case, the only difference between convolution and correlation is in the sign of the argument of $g$, which establishes whether or not $g$ is rotated around the origin.

- If $f$ is of the form $f(x,y) = f_1(x) \cdot f_2(y)$, then the following identity holds

$$\begin{aligned} f(x,y) &= f_i(x) \cdot f_2(y) \\ &= \left[ f_1(x) \cdot \delta(y) \right] *_{xy} \left[ \delta(x) \cdot f_2(y) \right] \end{aligned} \qquad (1.51)$$

- This way of expressing a function of separable variables is illustrated in Fig. 2 for $f(x,y) = \sin\frac{x}{a} \cdot \sin\frac{y}{b}$.

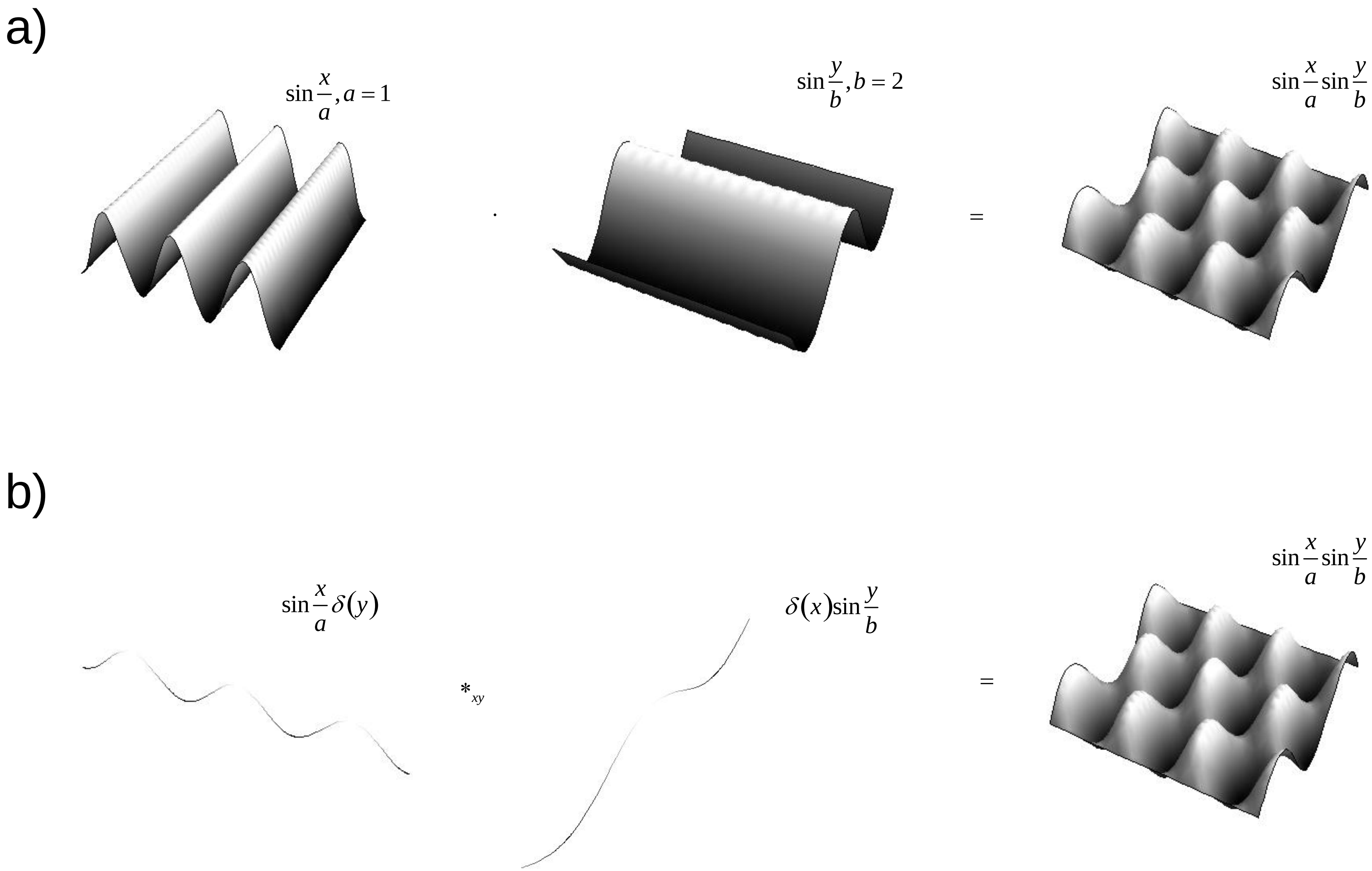



*Figure 2. Expressing sin(x/a)sin(y/b) as a product (a) and as a convolution (b).*

## 2.3. Theorems specific to two-dimensional functions

- *Shear theorem.* If $f(x, y)$ is sheared then its transform is sheared to the same degree in the perpendicular direction.

$$f(x+by, y) \rightarrow \tilde{f}(k_x, k_y - bk_x) \tag{1.52}$$

- *Proof:* Let us change variables to

$$\begin{aligned} u &= x+by \\ v &= y \end{aligned} \tag{1.53}$$

- The Fourier transform of the *sheared* function is

$$\begin{aligned}\tilde{f}(k_x,k_y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+by,y)\cdot e^{-i(k_x\cdot x+k_y\cdot y)}dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\cdot e^{-i\left[k_x(u-bv)+k_y\cdot v\right]}dudv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\cdot e^{-i\left[k_x\cdot u+v(k_y-bk_x)\right]}dudv \\ &= \tilde{f}(k_x,k_y-bk_x) \qquad (q.e.d.)\end{aligned} \tag{1.54}$$

- *Rotation theorem* If $f(x,y)$ is rotated in the $(x,y)$ plane then its Fourier transform is rotated in the $(k_x,k_y)$ plane by the same angle (and the same sense). The rotated coordinates are

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} \qquad (1.55)$$

- Thus the rotation theorem states

$$f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \rightarrow \tilde{f}(k_x\cos\theta - k_y\sin\theta, k_x\sin\theta + k_y\cos\theta) \qquad (1.56)$$

- The Fourier transform of the rotated function is

$$\tilde{f}_\theta(k_x,k_y) = \iint f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)e^{-i(k_x\cdot x + k_y\cdot y)}dxdy \qquad (1.57)$$

- *Proof:* Let us use a change of variables

$$
\begin{aligned}
u &= x\cos\theta - y\sin\theta \\
v &= x\sin\theta + y\cos\theta \\
x &= u\cos\theta + v\sin\theta \\
y &= -u\sin\theta + v\cos\theta
\end{aligned}
\qquad (1.58)
$$

- It follows that

$$
\begin{aligned}
dudv &= \begin{vmatrix} \dfrac{du}{dx} & \dfrac{du}{dy} \\ \dfrac{dv}{dx} & \dfrac{dv}{dy} \end{vmatrix} dxdy \\
&= dxdy
\end{aligned}
\qquad (1.59)
$$

- Equation 11 becomes

$$
\begin{aligned}
f_\theta(k_x,k_y) &= \iint f(u,v)\cdot e^{-i\left[k_x(u\cos\theta+v\sin\theta)+k_y(-u\sin\theta+v\cos\theta)\right]}dudv \\
&= \iint f(u,v)\cdot e^{-i\left[u(k_x\cos\theta-k_y\sin\theta)+v(k_x\sin\theta+k_y\cos\theta)\right]}dudv \qquad (1.60) \\
&= \tilde{f}(k_x\cos\theta-k_y\sin\theta, k_x\sin\theta+k_y\cos\theta) \qquad (q.e.d.)
\end{aligned}
$$

- *Affine theorem*. An affine transformation changes the vector $(x, y)$ into $(ax + by + c, dx + ey + f)$, i.e. it's a linear transformation followed by a shift.
- If an image $f(x, y)$ suffers an affine transformation, points that were *collinear* remain collinear. Further, ratios of distances along a line do not change upon transformation. The following property exists for the Fourier transform of an affine-transformed function

$$f(ax + by + c, dx + ey + f) \to e^{i\frac{(ec-bf)k_x+(af-cd)k_y}{ae-bd}} \cdot \tilde{f}\left(\frac{ek_x - dk_y}{ae - bd}, \frac{-bk_x + ak_y}{ae - bd}\right)$$

(1.61)

- *Proof:* Left as exercise (hint: make use of the shift, similarity and shear theorems)

## 2.4. Generalization of 1D theorems

- *Central ordinate theorem*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \tilde{f}(0,0) \qquad (1.62)$$

- *Shift theorem*

$$f(x-a, y-b) \rightarrow e^{-i(k_x \cdot a + k_y \cdot b)} \cdot \tilde{f}(k_x, k_y) \qquad (1.63)$$

- *Similarity theorem*

$$f(ax, by) \rightarrow \frac{1}{|ab|} \cdot \tilde{f}\left(\frac{k_x}{a}, \frac{k_y}{b}\right) \qquad (1.64)$$

- *Convolution theorem*

$$f(x,y) *_{xy} g(x,y) \rightarrow \tilde{f}(k_x, k_y) \cdot \tilde{g}(k_x, k_y) \qquad (1.65)$$

- *Correlation theorem*

$$f(x,y) \otimes_{xy} g(x,y) \rightarrow \tilde{f}(k_x, k_y) \cdot \tilde{g}^*(k_x, k_y) \qquad (1.66)$$

- *Modulation theorem*

$$f(x,y)\cos bx \rightarrow \frac{1}{2}\tilde{f}(k_x+b,k_y)+\frac{1}{2}\tilde{f}(k_x-b,k_y) \quad (1.67)$$

- *Parseval's theorem*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x,y)|^2\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(k_x,k_y)|^2\,dk_x dk_y \quad (1.68)$$

- *Differentiation properties*

$$\left(\frac{\partial}{\partial x}\right)^m\left(\frac{\partial}{\partial y}\right)^n f(x,y) \rightarrow (ik_x)^m(ik_y)^n\,\tilde{f}(k_x,k_y) \quad (1.69)$$

$$\left[\left(\frac{\partial}{\partial x}\right)^m+\left(\frac{\partial}{\partial y}\right)^n\right]f(x,y) \rightarrow \left[(ik_x)^m+(ik_y)^n\right]\tilde{f}(k_x,k_y) \qquad (1.70)$$

- *First moments*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy = i\frac{\partial\tilde{f}}{\partial k_x}(0,0)$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy = i\frac{\partial\tilde{f}}{\partial k_y}(0,0) \qquad (1.71)$$

- *Center of gravity*

$$\langle x\rangle = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy} = i\frac{\partial\tilde{f}/\partial k_x\,(0,0)}{\tilde{f}(0,0)}$$
$$\langle y\rangle = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy} = i\frac{\partial\tilde{f}/\partial k_y\,(0,0)}{\tilde{f}(0,0)} \qquad (1.72)$$

- *Second moments*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x,y)dxdy = -\frac{\partial^2 \tilde{f}}{\partial {k_x}^2}(0,0)$$

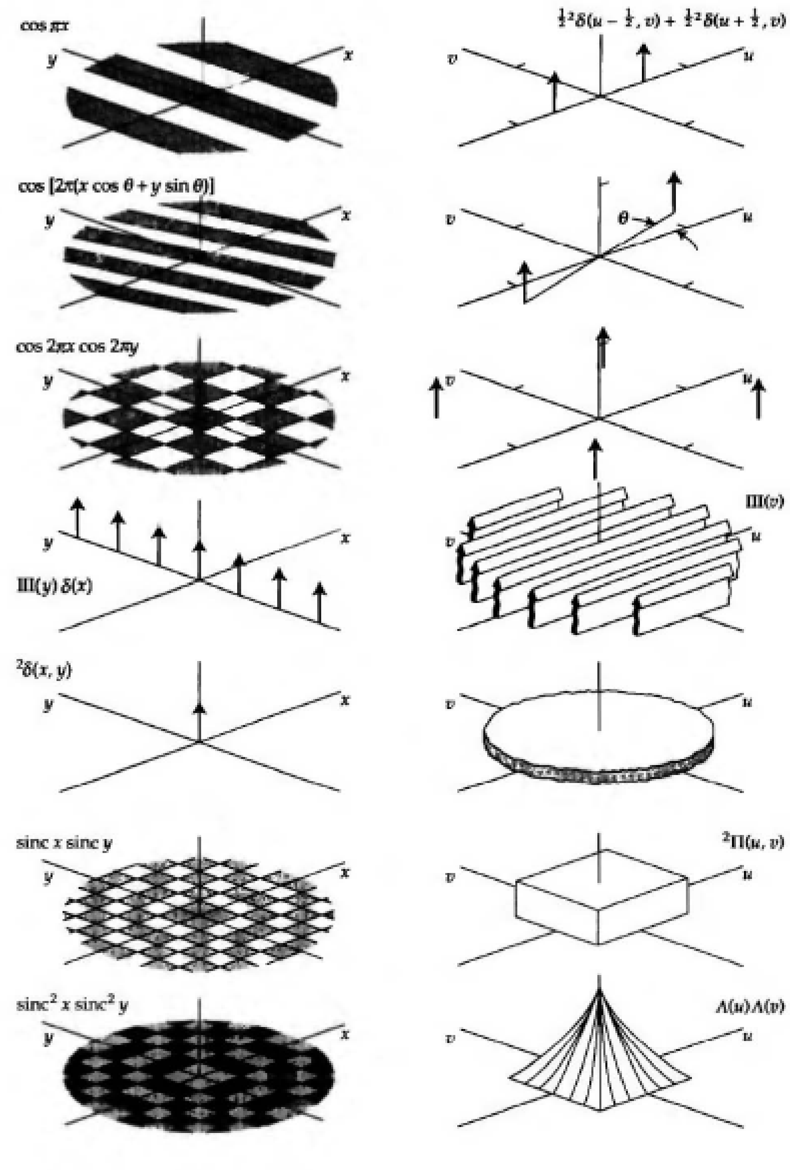
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x,y)dxdy = -\frac{\partial^2 \tilde{f}}{\partial {k_y}^2}(0,0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = -\frac{\partial^2 \tilde{f}}{\partial k_x \partial k_y}(0,0) \qquad (1.73)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(x^2+y^2\right) f(x,y)dxdy = -\left[\frac{\partial^2 \tilde{f}}{\partial {k_x}^2}(0,0) + \frac{\partial^2 \tilde{f}}{\partial {k_y}^2}(0,0)\right]$$

- *Equivalent width*

$$\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy}{f(0,0)} = \frac{\tilde{f}(0,0)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(k_x,k_y)dk_x dk_y} \qquad (1.74)$$



cos πx
y
x
½ ²δ(u − ½, v) + ½ ²δ(u + ½, v)
v
u
cos [2π(x cos θ + y sin θ)]
y
x
v
θ
u
cos 2πx cos 2πy
y
x
v
u
y
x
III(y) δ(x)
III(v)
v
u
²δ(x, y)
y
x
v
u
sinc x sinc y
y
x
²Π(u, v)
v
u
sinc² x sinc² y
y
x
Λ(u)Λ(v)
v
u

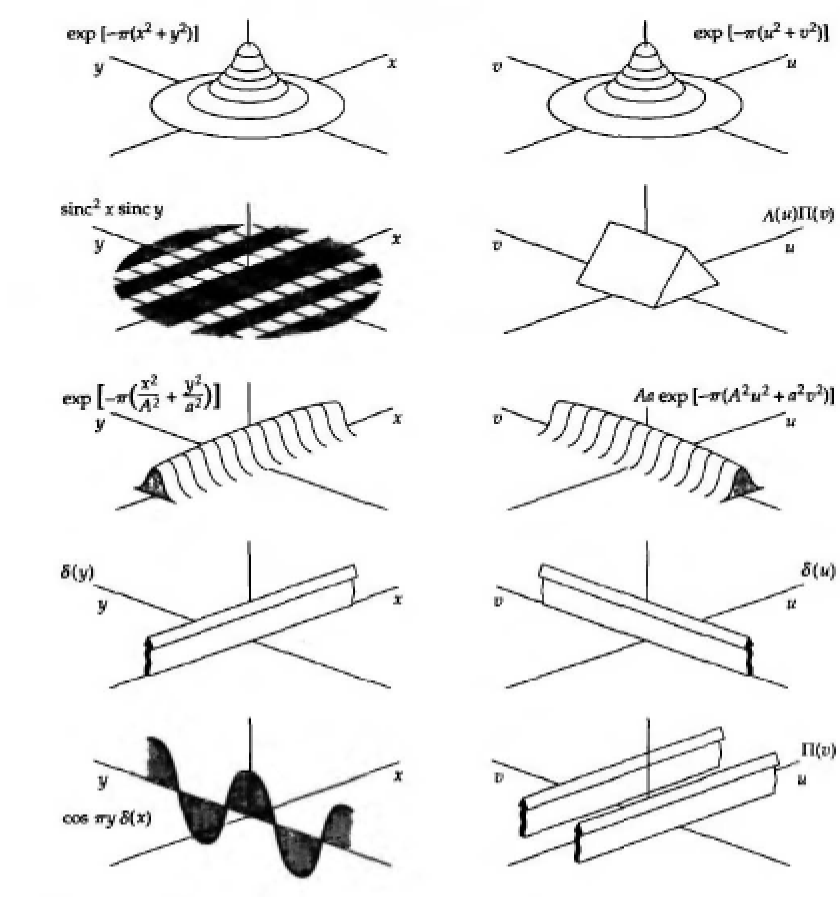



*Figure 3. Examples of 2D Fourier transforms (Bracewell).*

### *2.5. The Hankel transform*

- Many optical systems exhibit circular symmetry.
- Light emitted in 2D by a point source exhibits this symmetry. This problem simplifies significantly as the only non-trivial variable is the radial coordinate. Changing from Cartesian to polar coordinates, we obtain

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}\frac{y}{x} \end{aligned} \tag{1.75}$$

- Using the polar representation of the Fourier domain, we have

$$k_x = k\cos\theta'$$
$$k_y = k\sin\theta'$$
$$k = \sqrt{k_x{}^2 + k_y{}^2} \qquad (1.76)$$
$$\theta' = \tan^{-1}\frac{k_y}{k_x}$$

- From the rotation theorem, if a function is circularly symmetric, i.e. $f(x,y) = f(r)$, then its Fourier transform is also circularly symmetric, $f(k_x,k_y) = f(k)$.

- The Fourier transform is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot e^{-i\left(k_x\cdot x + k_y\cdot y\right)}dxdy = \int_{0}^{\infty}\int_{0}^{2\pi} f(r)\cdot e^{-ikr\cdot\cos(\theta-\theta')}rdrd\theta$$
$$= \int_{0}^{\infty} f(r)\left[\int_{0}^{2\pi} e^{-ikr\cdot\cos\theta}d\theta\right]rdr. \qquad (1.77)$$

- The integral in Eq. 34 does not depend on $\theta'$, as expected for circular symmetry. The integral over $\theta$ defines the Bessel function of *zeroth order* and *first kind*,

$$J_0(kr) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikr\cdot\cos\theta} d\theta \tag{1.78}$$

- Thus, the resulting Fourier relationships become

$$\begin{aligned} \tilde{f}(k) &= 2\pi\int_0^{\infty} f(r)J_0(kr)rdr \\ f(r) &= 2\pi\int_0^{\infty} \tilde{f}(k)J_0(kr)kdk \end{aligned} \tag{1.79}$$

- Equations 36a-b define a *Hankel transform* relationship (of zeroth order) between $f$ and $\tilde{f}$. Thus, because of the circular symmetry, the 2D Fourier transfer reduces to a 1D integral, where the $e^{i\mathbf{k}\cdot\mathbf{r}}$ kernel is replaced by $J_0(qr)$. Figure 4 illustrates the behavior of Bessel functions of orders $n = 0,1,2$.

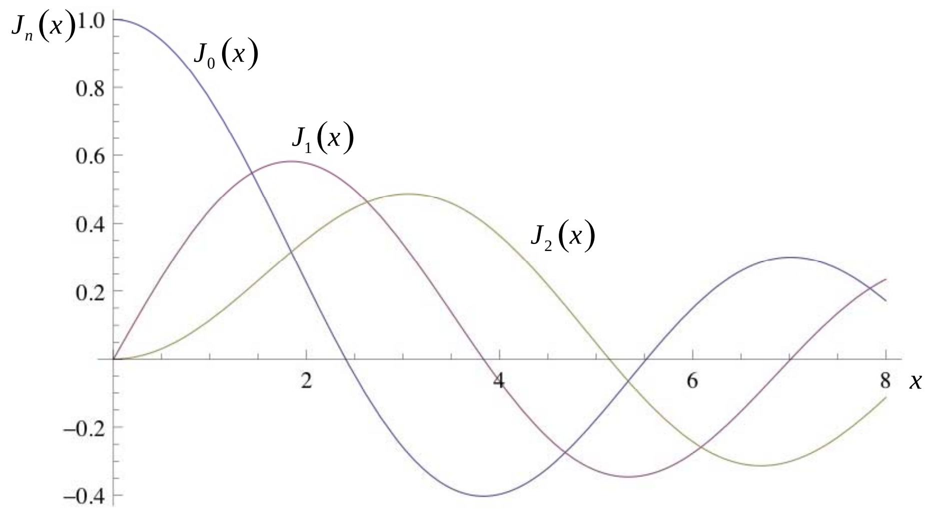



*Figure 4. Bessel functions of various orders.*

- Some useful identities for Bessel functions of first kind are

$$J_n(x) = \frac{1}{2\pi i^n} \cdot \int_0^{2\pi} e^{ix\cos\theta} \cdot e^{in\theta} d\theta$$

$$\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x) \tag{1.80}$$

$$\int_0^{\infty} J_n(x) dx = 1$$

| $f(r)$ | $\tilde{f}(k)$ |
|---|---|
| $\delta(x,y)=\dfrac{\delta(r)}{\pi r}$ | $1$ |
| $\delta(r-a)$ | $2\pi a J_0(ka)$ |
| $\Pi\left(\dfrac{r}{2a}\right)$ | $2\pi\dfrac{J_1(ka)}{k}$ |
| $\dfrac{1}{r}$ | $\dfrac{1}{k}$ |
| $e^{-\alpha r}$ | $\dfrac{2\pi\alpha}{\left(k^2+\alpha^2\right)^{3/2}}$ |
| $\dfrac{e^{-\alpha r}}{r}$ | $\dfrac{2\pi\alpha}{\left(k^2+\alpha^2\right)^{1/2}}$ |

| | |
|---|---|
| $e^{-\alpha r^2}$ | $e^{-\frac{k^2}{4\alpha}}$ |

*Table 1. Common Hankel transform pairs*

- The Hankel transform satisfies some important theorems, which are analogous to those of 1D and 2D Fourier transforms.
- *Central ordinate theorem*

$$\begin{aligned}\tilde{f}(0) &= 2\pi\int_0^{\infty}\tilde{f}(r)J_0(0)rdr \\ &= 2\pi\int_0^{\infty}f(r)rdr\end{aligned} \qquad (1.81)$$

- *Shift theorem*
  - o The circular symmetry is destroyed upon a shift in origin; Hankel transform does not apply.
- *Similarity theorem*

$$f(ar) \to \frac{1}{a^2}\tilde{f}\left(\frac{k}{a}\right) \tag{1.82}$$

- *Convolution theorem*

$$\int_0^{\infty}\int_0^{2\pi} f(r')g(r)r'dr'd\theta \rightarrow \tilde{f}(k)\cdot\tilde{g}(k) \qquad (1.83)$$

$$\left(\rho^2 = r^2 + r'^2 - 2rr'\cos\theta\right)$$

- *Parseval's theorem*

$$\int_0^{\infty}\left|f(r)\right|^2 rdr = \frac{1}{(2\pi)^2}\int_0^{\infty}\left|\tilde{f}(k)\right|^2 kdk \qquad (1.84)$$

- *Laplacian*

$$\nabla^2 f = \frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} \rightarrow -k^2\tilde{f}(k) \qquad (1.85)$$

- *Second moment*

$$\int_0^{\infty} r^2 f(r)rdr = \frac{\tilde{f}''(0)}{-4\pi^3} \qquad (1.86)$$

- *Equivalent width*

$$\frac{2\pi\int_0^{\infty} f(r)rdr}{f(0)} = \frac{\tilde{f}}{\frac{1}{2\pi}\int_0^{\infty} \tilde{f}(k)kdk} \qquad (1.87)$$

## 3. THE 3D FOURIER TRANSFROM

### 3.1. Definition

- The Fourier pairs naturally extend to 3D functions as

$$f(x,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}\left(k_x,k_y,k_z\right)\cdot e^{i(k_x\cdot x+k_y\cdot y+k_z\cdot z)}dk_x dk_y dk_z$$

(1.88)

$$\tilde{f}(k_x,k_y,k_z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y,z)\cdot e^{-i(k_x\cdot x+k_y\cdot y+k_z\cdot z)}dxdydz$$

- Below we discuss the 3D Fourier transform in *cylindrical* and *spherical coordinates.*

## 3.2. Cylindrical coordinates

- In this case,

$$f(x,y,z)=g(r,\theta,z)$$
$$\tilde{f}(k_x,k_y,k_z)=\tilde{g}(k_\perp,\theta',k_z), \qquad (1.89)$$

- where

$$x+iy=r\cdot e^{i\theta}$$
$$k_x+ik_y=k_\perp\cdot e^{i\theta'} \qquad (1.90)$$

- The functions $g$ and $\tilde{g}$ are related by

$$g(r,\theta,z)=\int_{-\infty}^{\infty}\int_{0}^{2\pi}\int_{0}^{\infty}\tilde{g}(k_\perp,\theta',k_z)\cdot e^{i\left[k_\perp r\cos(\theta-\theta')+k_z\cdot z\right]}dk_\perp d\theta' dk_z$$
$$\tilde{g}(k_\perp,\theta',k_z)=\int_{-\infty}^{\infty}\int_{0}^{2\pi}\int_{0}^{\infty}g(r,\theta,z)\cdot e^{-i\left[k_\perp r\cos(\theta-\theta')+k_z\cdot z\right]}dr d\theta dz \qquad (1.91)$$

- Under *circular symmetry*, i.e. $f$ independent of $\theta$, and $\tilde{f}$ independent of $\theta'$,

$$f(x,y,z) = h(r,z)$$
$$\tilde{f}(k_x,k_y,k_z) = \tilde{h}(k_\perp,k_z) \qquad (1.92)$$

- Equations 48a-b become

$$h(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\tilde{h}(k_\perp,k_z)\cdot J_0(k_\perp\cdot r)\cdot e^{ik_z\cdot z}\cdot k_\perp dk_\perp dk_z$$
$$\tilde{h}(k_\perp,k_z) = 2\pi\int_{-\infty}^{\infty}\int_{0}^{\infty}h(r,z)\cdot J_0(k_\perp\cdot r)\cdot e^{-ik_z\cdot z}\cdot r dr dz \qquad (1.93)$$

- The integral in Eq. 50a represents a 1D Fourier transform along z of the Hankel transform with respect to r.

- If the problem has *cylindrical symmetry*, we have

$$f(x,y,z) = p(r)$$
$$\tilde{p}(k_x,k_y,k_z) = \tilde{p}(k_\perp)\cdot\delta(k_z) \qquad (1.94)$$

- The Fourier transform simplifies to

$$
\begin{aligned}
p(r) &= \frac{1}{2\pi}\int_0^\infty \tilde{p}(k_\perp)\cdot J_0(k_\perp r)k_\perp dk_\perp \\
\tilde{p}(k_\perp) &= 2\pi\int_0^\infty p(r)\cdot J_0(k_\perp r)rdr
\end{aligned}
\qquad (1.95)
$$

- This transformation is important for studying paraxial propagation of light, where the z-axis propagation only contributes a phase shift $kz$.

## 3.3. Spherical coordinates

- In spherical coordinates,

$$f\left(x,y,z\right)=g\left(r,\theta,\phi\right)$$
$$\tilde{f}\left(k_x,k_y,k_z\right)=\tilde{g}(k,\theta',\phi'), \tag{1.96}$$

- The change of coordinates follows

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta$$
$$k_x=k\sin\theta'\cos\phi',\ k_y=k\sin\theta'\sin\phi',\ k_z=k\cos\theta' \tag{1.97}$$

- The Fourier integrals become

$$g\left(r,\theta,\phi\right)=\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\tilde{g}\left(k,\theta',\phi'\right)\cdot e^{i\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\left(\phi-\phi'\right)\right]}\cdot k^2\sin\theta' dkd\theta'd\phi'$$
$$\tilde{g}\left(k,\theta',\phi'\right)=\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}g\left(r,\theta,\phi\right)\cdot e^{-i\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\left(\phi-\phi'\right)\right]}\cdot r^2\sin\theta drd\theta d\phi$$

(1.98)

- Under *circular symmetry*, i.e. $f(x,y,z)$ is independent of $\phi$, we have

$$\begin{aligned} f(x,y,z) &= g(r,\theta) \\ \tilde{f}(k_x,k_y,k_z) &= \tilde{g}(k,\theta') \end{aligned} \qquad (1.99)$$

- The Fourier transforms are

$$\begin{aligned} g(r,\theta) &= 2\pi\int_0^\infty\int_0^\pi \tilde{g}(k,\theta')J_0(kr\sin\theta\cdot\sin\theta')\cdot e^{ikr\cos\theta\cos\theta'}r^2\sin\theta'\,drd\theta' \\ \tilde{g}(k,\theta') &= 2\pi\int_0^\infty\int_0^\pi g(r,\theta)J_0(kr\sin\theta\cdot\sin\theta')\cdot e^{-ikr\cos\theta\cos\theta'}r^2\sin\theta\,drd\theta \end{aligned} \qquad (1.100)$$

- With *spherical symmetry*, we have

$$\begin{aligned} f(x,y,z) &= h(r) \\ \tilde{f}(k_x,k_y,k_z) &= \tilde{h}(k) \end{aligned} \qquad (1.101)$$

- In this case, the integrals reduce to

$$h(r) = \frac{1}{2\pi^2}\int_0^\infty \tilde{h}(k)\,\mathrm{sinc}(kr)\,k^2 dk$$

$$\tilde{h}(k) = 4\pi\int_0^\infty h(r)\,\mathrm{sinc}(kr)\,r^2 dr \qquad (1.102)$$

- A few examples of 3D Fourier pairs are shown in Table 2.

| $f(x,y,z)$ | $\tilde{f}(k_x,k_y,k_z)$ |
|---|---|
| $\delta(x-a,y-b,z-c)$ | $e^{i(k_x \cdot a + k_y \cdot b + k_z \cdot c)}$ |
| $\Pi(x,y,z)$ (cube) | $\frac{1}{(2\pi)^3}\sin\left(\frac{k_x}{2\pi}\right)\cdot\sin\left(\frac{k_y}{2\pi}\right)\cdot\sin\left(\frac{k_z}{2\pi}\right)$ |
| $\Pi(x,y)$ (bar) | $\frac{1}{(2\pi)^2}\sin\left(\frac{k_x}{2\pi}\right)\cdot\sin\left(\frac{k_y}{2\pi}\right)\cdot\delta(k_z)$ |

$$\Pi(x)$$

(slab)

$$\frac{1}{2\pi}\cdot\sin\left(\frac{k_x}{2\pi}\right)\cdot\delta(k_y)\cdot\delta(k_z)$$

$$\Pi\left(\frac{r}{2}\right) \text{ (ball)}$$

$$\frac{\sin k - k\cdot\cos k}{2k^3}$$

$$\Pi(x)\Pi\left(\sqrt{y^2+z^2}\right)$$

(disk)

$$\frac{1}{(2\pi)^3}\sin\left(\frac{k_x}{2\pi}\right)\cdot\frac{J_1\left(\sqrt{k_y^{\ 2}+k_z^{\ 2}}\Big/2\right)}{\sqrt{k_y^{\ 2}+k_z^{\ 2}}\Big/\pi}$$

$$\frac{e^{-\alpha r}}{4\pi r}$$

$$\frac{1}{\alpha^2-k^2}$$

$$\frac{e^{-r/R}}{\frac{4}{3}\pi R^3}$$

$$\frac{1}{(2\pi)^3}\frac{6}{\left(1+k^2R^2\right)^2}$$

$$e^{-\frac{\alpha r^2}{2}}$$

$$e^{-\frac{k^2}{2\alpha}}$$