

## 5. GAUSSIAN BEAMS

### 5.1. Solution to the wave equation in Cartesian coordinates

Recall the Helmholtz equation for a (scalar) field  $U$  in rectangular coordinates

$$\nabla^2 U(\mathbf{r}, \omega) + \beta^2(\mathbf{r}, \omega)U(\mathbf{r}, \omega) = 0, \quad (5.1)$$

Where  $\beta$  is the *propagation constant* (material property), defined as

$$\begin{aligned} \beta^2(\mathbf{r}, \omega) &= \omega^2 \mu \epsilon(\mathbf{r}, \omega) - i\omega \mu \sigma(\mathbf{r}, \omega) \\ &= n^2(\mathbf{r}, \omega) \frac{\omega^2}{c^2} - i\omega \mu \sigma(\mathbf{r}, \omega) \end{aligned} \quad (5.2)$$

Assuming lossless medium ( $\sigma = 0$ ) and decoupling the vacuum contribution ( $n = 1$ ) from  $\beta(\mathbf{r}, \omega)$ , we re-write Eq. 2 to explicitly show the driving term, namely

$$\nabla^2 U(\mathbf{r}, \omega) + k_0^2 U(\mathbf{r}, \omega) = -k_0^2 [n^2(\mathbf{r}, \omega) - 1]U(\mathbf{r}, \omega), \quad (5.3)$$

where  $k_0 = \omega/c$ . Note that Eq. 3 preserves the generality of the Helmholtz equation. The Green's function,  $h$ , (an impulse response) is obtained by setting the driving term (the right hand side) to a delta function,

$$\begin{aligned} \nabla^2 h(\mathbf{r}, \omega) + k_0^2 h(\mathbf{r}, \omega) &= -^{(3)}\delta(\mathbf{r}) \\ &= -\delta(x)\delta(y)\delta(z) \end{aligned} \quad (5.4)$$

In order to solve this equation, we take the Fourier transform with respect to  $\mathbf{r}$ ,

$$-k^2 \tilde{h}(\mathbf{k}, \omega) + k_0^2 \tilde{h}(\mathbf{k}, \omega) = -1, \quad (5.5)$$

where  $k^2 = \mathbf{k} \cdot \mathbf{k}$ .

For propagation in 1D, say along x, we have

$$\begin{aligned} \tilde{h}(k_x, \omega) &= \frac{1}{k_x^2 - k_0^2} \\ &= \frac{1}{2k_{0x}} \left[ \frac{1}{k_x - k_0} - \frac{1}{k_x + k_0} \right] \end{aligned} \quad (5.6)$$

To calculate the Fourier transform of Eq. 6, we invoke the shift theorem and the Fourier transform of function  $1/k$ ,

$$\begin{aligned} f(k-a) &\rightarrow e^{iax} \cdot \tilde{f}(x) \\ \frac{1}{k} &\rightarrow i \operatorname{sign}(x) \end{aligned} \quad (5.7)$$

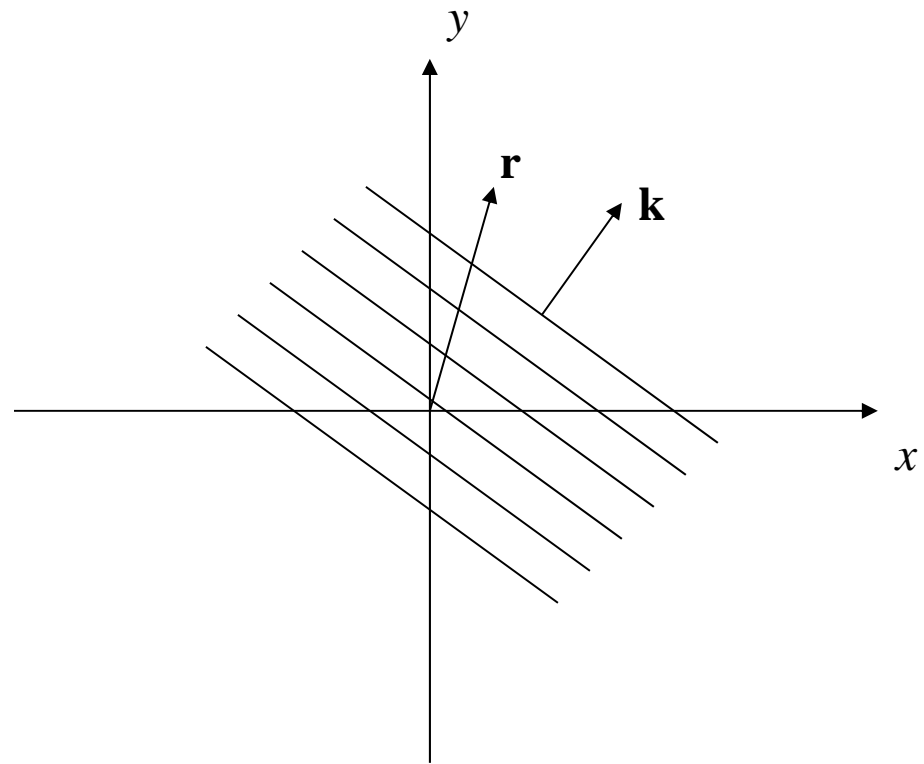
Thus we obtain as the final solution

$$h(x, \omega) = \frac{1}{ik_0} \cdot e^{ik_0x}, \quad x > 0 \quad (5.8)$$

The procedure applies to all three dimensions, such that the 3D solution reads

$$h(\mathbf{r}, \omega) \propto e^{ik_0 \hat{\mathbf{k}} \cdot \mathbf{r}}, \quad (5.9)$$

where  $\hat{\mathbf{k}}$  is the unit vector,  $\hat{\mathbf{k}} = \mathbf{k} / k$ . Equation 9 describes the well known *plane wave* solution, which is characterized by the absence of amplitude modulation upon propagation. This is an infinitely broad wavefront that propagate along direction  $\hat{\mathbf{k}}_0$  (Figure 1).



*Figure 5-1. Plane wave.*

## 5.2. Solution of the wave equation in spherical coordinates

For light propagation with spherical symmetry, such as emission from a point source in free space, the problem becomes one dimensional, with the radial coordinate as the only variable,

$$\begin{aligned}
 h(\mathbf{r}, \omega) &= U(r, \omega) \\
 \tilde{h}(\mathbf{k}, \omega) &= \tilde{U}(k, \omega) \\
 {}^{(3)}\delta(\mathbf{r}) &= \frac{1}{4\pi r^2} {}^{(1)}\delta(r) \\
 {}^{(3)}\delta(\mathbf{r}) &\rightarrow 1
 \end{aligned} \tag{5.10}$$

Recall that the Fourier transform pair is defined in this case as

$$\begin{aligned}
 h(r) &\propto \int_0^\infty \tilde{h}(k) \cdot \text{sinc}(kr) k^2 dk \\
 \tilde{h}(k) &\propto \int_0^\infty h(r) \cdot \text{sinc}(kr) r^2 dr
 \end{aligned} \tag{5.11}$$

The Fourier properties of  ${}^{(3)}\delta(r)$  and  $\nabla^2$  extend naturally to the spherically symmetric case as

$$\begin{aligned}
 {}^{(3)}\delta(r) &\rightarrow 1 \\
 \nabla^2 &\rightarrow -k^2
 \end{aligned} \tag{5.12}$$

Thus, by Fourier transforming Eq. 4, we obtain the frequency domain solution,

$$\tilde{h}(k, \omega) = \frac{1}{k^2 - k_0^2} \tag{5.13}$$

Not surprisingly, the frequency domain solutions for the Cartesian and spherical coordinates (Eqs. 6 and 13, respectively) look quite similar, except that the former depends on one component of the wave vector and the latter on the modulus of the wave vector. The solution in the spatial domain becomes

$$\begin{aligned}
 h(r, \omega) &= \int_0^{\infty} \frac{1}{k^2 - k_0^2} \cdot \frac{\sin(kr)}{kr} \cdot k^2 dk \\
 &= \frac{1}{2r} \int_0^{\infty} \left( \frac{1}{k - k_0} + \frac{1}{k + k_0} \right) \frac{e^{ikr} - e^{-ikr}}{2i} dk \\
 &= \frac{1}{4ir} \int_0^{\infty} \frac{e^{ikr}}{k - k_0} dk.
 \end{aligned} \tag{5.14}$$

We recognize in Eq. 15 the Fourier transform of a shifted  $1/k$  function, which we encountered earlier (Eqs. 7). Thus, evaluating this Fourier transform, we finally obtain Green's function for propagation from a point source,

$$h(x, \omega) \propto \frac{e^{ikr}}{r}, \quad r > 0 \tag{5.15}$$

This solution defines a (outgoing) spherical wave.

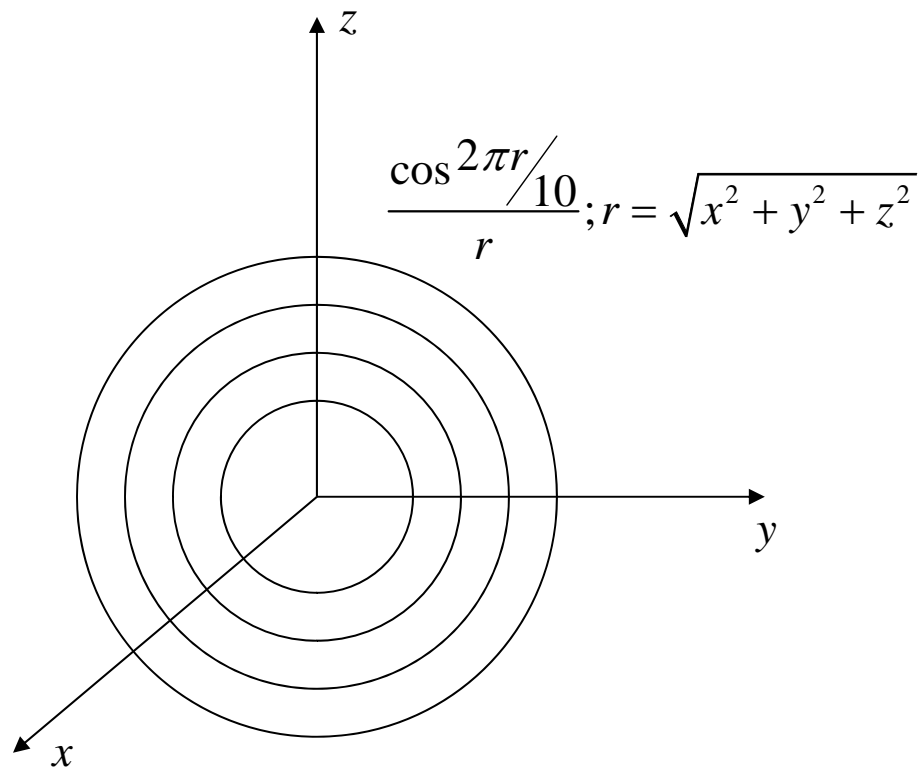


Figure 5-2. Spherical wave

### 5.3. Solution of the wave equation in cylindrical coordinates

Perhaps the most common situation from an experimental point of view is light propagation in cylindrical coordinates with radial symmetry, e.g. the  $z$  coordinate defines the optical axes and the radial coordinate,  $r$ , defines the transverse coordinate. The Fourier pair in this case can be written as

$$h(r, z) = \int_{-\infty}^{\infty} \int_0^{\infty} \tilde{h}(k_{\perp}, k_z) J_0(k_{\perp} \cdot r) \cdot e^{ik_z \cdot z} \cdot k_{\perp} \cdot dk_{\perp} dk_z$$

$$\tilde{h}(k_{\perp}, k_z) = \int_{-\infty}^{\infty} \int_0^{\infty} h(r, z) J_0(k_{\perp} \cdot r) \cdot e^{-ik_z \cdot z} \cdot r \cdot dr dz$$
(5.16)

Due to the angular symmetry in both the spherical and cylindrical coordinates, the spatial frequency domain solution of the Helmholtz equation is identical in form with Eq. 13 and depends only on the modulus of the wave vector (Eq. 13)

$$\tilde{h}(k, \omega) = \frac{1}{k^2 - k_0^2},$$

$$k^2 = k_{\perp}^2 + k_z^2$$
(5.17)

The solution  $\tilde{U}(k, \omega)$  was obtained following the procedure used in the spherically symmetric case, i.e. replacing the source term with  $^{(3)}\delta(r, z)$  and using the property  $\nabla^2 \rightarrow -k^2$  that holds for cylindrical coordinates with circular symmetry. We now Fourier transform Eq. 13 back to the spatial domain,

$$h(r, z) = \int_{-\infty}^{\infty} \int_0^{\infty} \frac{1}{k_z^2 - (k_0^2 - k_{\perp}^2)} \cdot J_0(k_{\perp} \cdot r) \cdot e^{ik_z \cdot z} \cdot k_{\perp} \cdot dk_{\perp} dk_z$$
(5.18)

Interestingly, Eq. 18 can be interpreted as the Hankel transform (over  $k_{\perp}$ ) of a 1D Fourier transform (over  $k_z$ ). In evaluating the integral over  $k_z$  first, we recover the 1D case discussed in Section 5.1, i.e. the solution in Eq. 8, where we replace  $k_0$  with  $\sqrt{k_0^2 - k_{\perp}^2}$  and  $x$  with  $z$ . Thus the remaining integral over  $k_{\perp}$  becomes

$$h(r, z) = \int_0^{\infty} \frac{e^{iqz}}{q} \cdot J_0(k_{\perp} \cdot r) \cdot k_{\perp} dk_{\perp}, \quad (5.19)$$

$$q = \sqrt{k_0^2 - k_{\perp}^2}$$

Equation 19a can be regarded as the exact Green's function in cylindrical coordinates with circular symmetry. The integral can be evaluated analytically in the limit of low transverse frequencies, i.e.  $k_{\perp} \ll k_0$ . In this case we expand  $\sqrt{k_0^2 - k_{\perp}^2}$  in Taylor series to obtain

$$\frac{e^{iqz}}{q} \simeq \frac{e^{ik_0z} \left[ 1 - \frac{k_{\perp}^2}{2k_0^2} \right]}{k_0}. \quad (5.20)$$

Note that while the amplitude term,  $1/q$ , varies slowly with  $k_{\perp}$ , such that the zeroth order term,  $k_0$ , is sufficient, the phase term is more sensitive, indicating that the second order term is necessary. Following this expansion, the Hankel transform remains to be calculated for a Gaussian phase function  $e^{-\frac{iz}{2k_0} k_{\perp}^2}$ . Here we recall that Hankel transforms of Gaussian functions can be easily calculated and follow a similar transformation rule with that of 1D functions,  $e^{-\frac{ix^2}{2b^2}} \rightarrow b^2 e^{-\frac{ib^2 k^2}{2}}$ . Finally, by combining Eqs. 20 and 19a, we obtain

$$h(r, z) \propto \frac{e^{ik_0z}}{z} \cdot e^{\frac{ik_0}{2z} r^2}. \quad (5.21)$$

This solution is characterized by a slow attenuation with distance,  $\propto \sqrt{z}$ , a plane wave-like phase delay vs.  $z$ , and a Gaussian transverse phase, of width that also grows as  $\sqrt{z}$ .

#### 5.4. Propagation of Gaussian beams

Often, experiments involve *light beams*. A light beam can be defined as a distribution of field that fulfills the approximation in Eq. 20, i.e. is characterized by a dominant wave vector component,  $k_z \gg k_x, k_y$ . A beam is, therefore, the spatial equivalent of *quasi-monochromatic light*, where the field is characterized by a dominant (temporal) frequency component.

A *Gaussian beam*, such as that delivered by a single (spatial) mode laser, exhibits a field distribution described by a Gaussian function in the transverse coordinate. The Green's function in Eq. 21 provides an efficient way to propagate this type of field. Let us consider a Gaussian beam that at  $z=0$  has the transverse distribution

$$U_0(r, 0) = \frac{1}{w_0} e^{-\frac{r^2}{w_0^2}}, \quad (5.22)$$

where  $w_0$  defines the minimum beam waist. Upon propagating over a distance  $z$ , the field becomes a convolution between  $U_0$  and  $h$ ,

$$U(r, z) = U_0(r, 0) * h(r, z) \quad (5.23)$$

We take the Fourier transform of Eq. 23, make use of the convolution theorem and use the approximation in Eq. 20 to obtain the frequency domain solution

$$\tilde{U}(k_\perp, z) = \frac{e^{ik_0 z}}{k_0} \cdot e^{\frac{izk_\perp^2}{2k_0}} \cdot e^{-\frac{w_0^2 k_\perp^2}{4}} \quad (5.24)$$

The field distribution, i.e. the spatial domain solution, is obtained by taking the inverse Hankel transform of  $\tilde{U}$ ,

$$U(r, z) = \int_0^\infty \tilde{U}(k_\perp, z) \cdot J_0(k_\perp r) k_\perp dk_\perp. \quad (5.25)$$

Thus, after propagation over distance  $z$ , the field remains Gaussian in the transverse direction and has the form

$$U(r, z) = \frac{e^{ik_0 z}}{c^2(z)} e^{-\frac{r^2}{c^2(z)}} \quad (5.26)$$

$$c^2(z) = w_0^2 + i \frac{2z}{k_0}$$

Note that the solution has a particularly simple form, thanks to the properties of Gaussian functions and their transforms. Interestingly, all the information regarding the beam change due to propagation is contained in function  $c(z)$ . This function can be rearranged as

$$c^2(z) = w_0^2 \left( 1 + i \frac{z}{z_0} \right) \quad (5.27)$$

$$z_0 = \frac{k_0 w_0^2}{2} = \frac{\pi w_0^2}{\lambda}$$

We can write explicitly the absolute value and phase of  $c^2(z)$ , i.e. express it in polar coordinates, as

$$c^2(z) = w^2(z) \cdot e^{i\psi(z)}, \quad (5.28)$$

where

$$w^2(z) = w_0^2 \left( 1 + \frac{z^2}{z_0^2} \right) \quad (5.29)$$

$$\psi(z) = \arg \left( \frac{z}{z_0} \right)$$

The distance  $z_0$  is known as the Rayleigh range, over which the beam waist increases by  $\sqrt{2}$ . Finally, we can re-write  $U(r, z)$  explicitly,

$$\begin{aligned}
U(r, z) &= \frac{1}{w(z)} \cdot e^{i \left[ k_0 z + \frac{z}{z_0} \frac{r^2}{w^2(z)} - \psi(z) \right]} \cdot e^{-\frac{r^2}{w^2(z)}} \\
&= A(r, z) \cdot e^{i\phi(r, z)}
\end{aligned} \tag{5.30}$$

Equation 5.30 represents the fundamental solution of the Gaussian beam propagation. It can be seen that both the broadening in  $r$  and propagation in  $z$  requires lowering the amplitude, such that the energy is conserved,

$$A(r, z) = \frac{1}{w_0 \sqrt{1 + \frac{z^2}{z_0^2}}} \cdot e^{-\frac{r^2}{w_0^2 \left(1 + \frac{z^2}{z_0^2}\right)}} \tag{5.31}$$

The new beam waist is, therefore,  $w(z) = w_0 \sqrt{1 + \frac{z^2}{z_0^2}}$ . The phase dependence reveals even more interesting features

$$\phi(r, z) = k_0 z + \frac{k_0 r^2}{2R(z)} - \arg\left(\frac{z}{z_0}\right) \tag{5.32}$$

Besides the plane wave phase delay,  $k_0 z$ , the wavefront radius of curvature  $R(z)$  induces additional phase shift, with  $R(z)$  defined as

$$R(z) = z \left( 1 + \frac{z_0^2}{z^2} \right) \tag{5.33}$$

Interestingly, the wavefront is centered a distance  $d(z) = 2z_0 / z$  behind the  $z=0$  plane (Fig. 3).

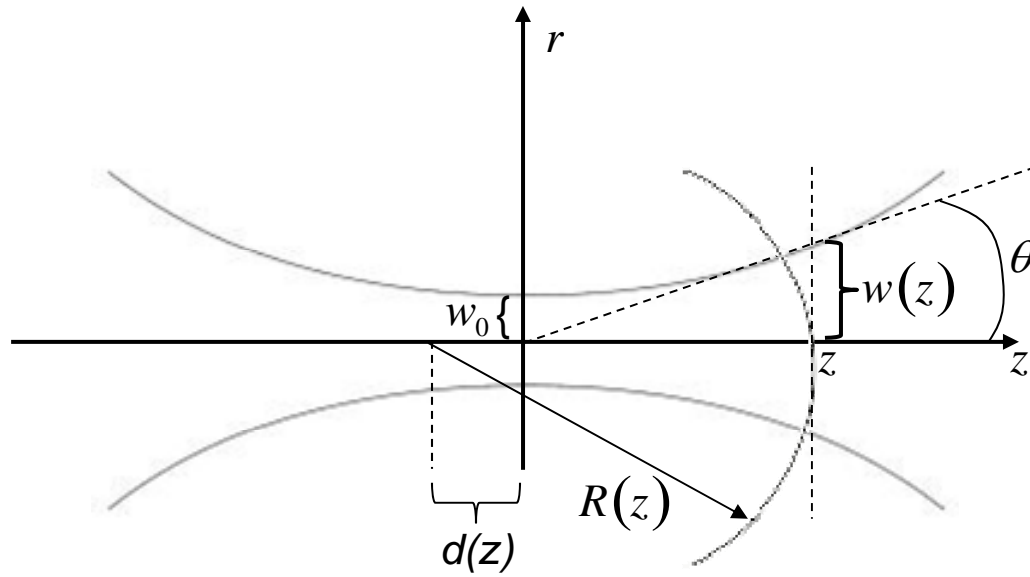


Figure 5-3. Propagation of Gaussian beams.

We can define a far field divergence angle,

$$\begin{aligned}
 \theta &= \lim_{z \rightarrow \infty} \frac{w(z)}{z} \\
 &= \frac{w_0}{z_0} \\
 &= \frac{\lambda}{\pi w_0}
 \end{aligned}
 \tag{5.34}$$

In far field,  $R(z) \approx z$ .

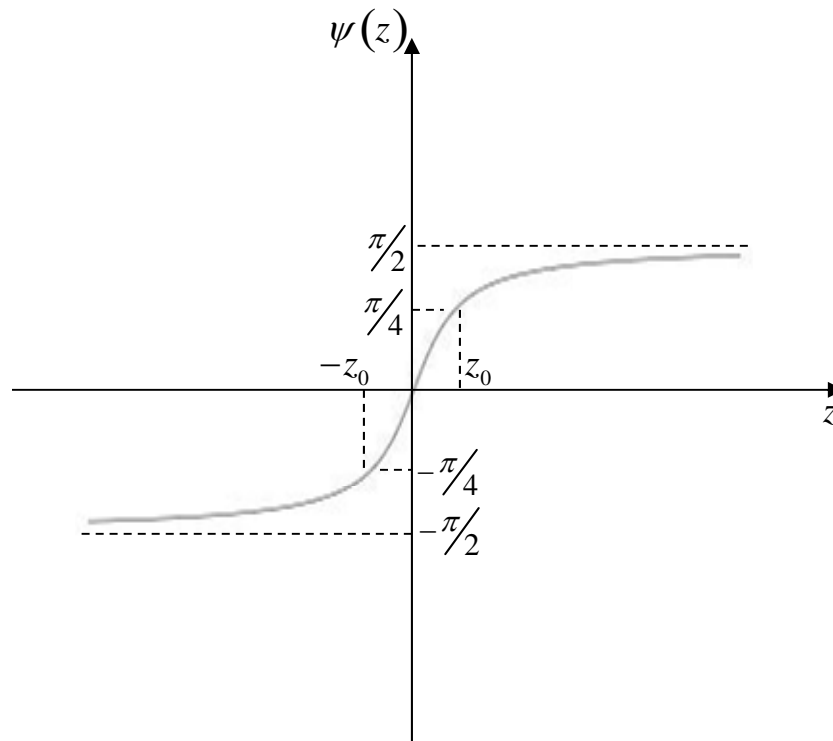


Figure 5-4. Gouy phase.

Perhaps the most surprising result is the phase delay due to the last term in Eq. 33,  $\psi(z) = \arg\left(\frac{z}{z_0}\right)$ . It can be seen that on propagating from  $z=0$  to  $\infty$ , the wavefront is advanced by  $\pi/2$ . Overall, from  $-\infty$  to  $\infty$ , there is a total phase shift of  $\pi$ . This peculiar

phase shift is referred to as *Gouy's phase*. This phase advancement contributes to the overall phase velocity  $v_p$  along the  $z$  axis, such that it exceeds the speed of light in vacuum,

$$v_p = \frac{\omega}{k_{eff}}, \quad (5.35)$$

where  $k_{eH}$  is the effective modulus of the wave vector, defined as

$$\begin{aligned} k_{eff} &= \frac{d}{dz} [k_0 z - \psi(z)] \\ &= k_0 - \frac{1/z_0}{1 + \frac{z^2}{z_0^2}} < k_0. \end{aligned} \quad (5.36)$$

Thus, the contribution of the Gouy phase is always to advance the phase and bring the phase velocity to superluminal values,  $v_p > c$ .

### 5.5 Matrix formalism for Gaussian beam propagation.

We discussed that in the far zone,  $z \gg z_0$ , we can define a diffraction angle,  $\theta = w/z$ . In this asymptotic regime, the propagation is well described by straight lines, which recovers geometrical optics. In the following we will develop a *matrix formalism* for retrieving the beam characteristics upon propagation.

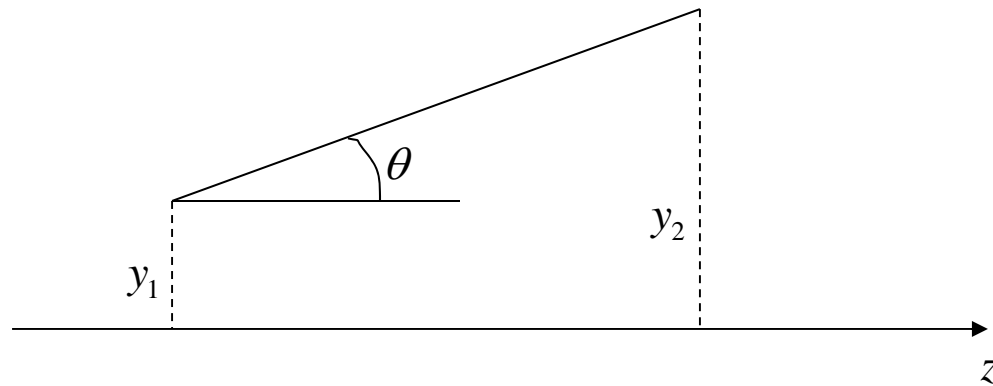


Figure 5-5. Geometrical limit of Gaussian beam propagation.

Let us find the transformation rules for the elevation  $y$  and angle  $\theta$  with the horizontal associated with the ray (Fig. 5). In particular, we will investigate propagation in free space (translation), refraction, and propagation through a thin lens.

#### 5.5.1 Translation.

During free space propagation, the angle stays constant and only the elevation changes,

$$\begin{aligned} y_2 &= y_1 + d\theta_1 \\ \theta_2 &= 0 \cdot y_1 + \theta_1 \end{aligned} \tag{5.37}$$

Equation 37 is, of course, valid for very small angles,  $\tan \theta \approx \theta$ . This transformation can be expressed in matrix form,

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = M_d \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}, M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \quad (5.38)$$

Equation 38 establishes the characteristic matrix for translation.

### 5.5.2 Refraction at dielectric interface.

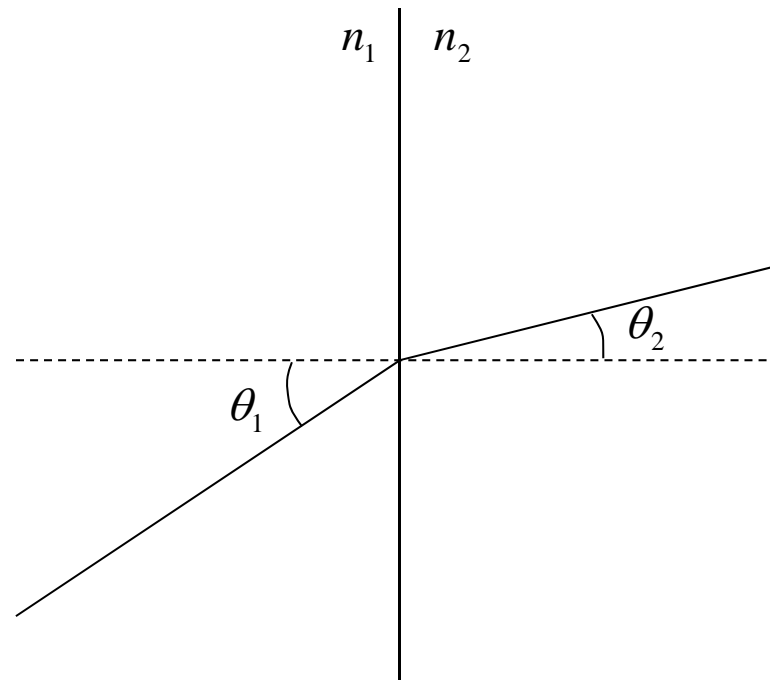


Figure 5-6. Refraction at a planar interface.

At the interface between two dielectric media, only the angle changes, according to Snell's law,

$$\begin{aligned}
 y_2 &= y_1 \\
 \theta_2 &= \frac{n_1 \theta_1}{n_2}
 \end{aligned}
 \tag{5.39}$$

Thus, the *refraction matrix* has the form

$$M_R = \begin{pmatrix} 1 & 0 \\ 1 & \frac{n_1}{n_2} \end{pmatrix}
 \tag{5.40}$$

5.5.3 Propagation through a thin lens.

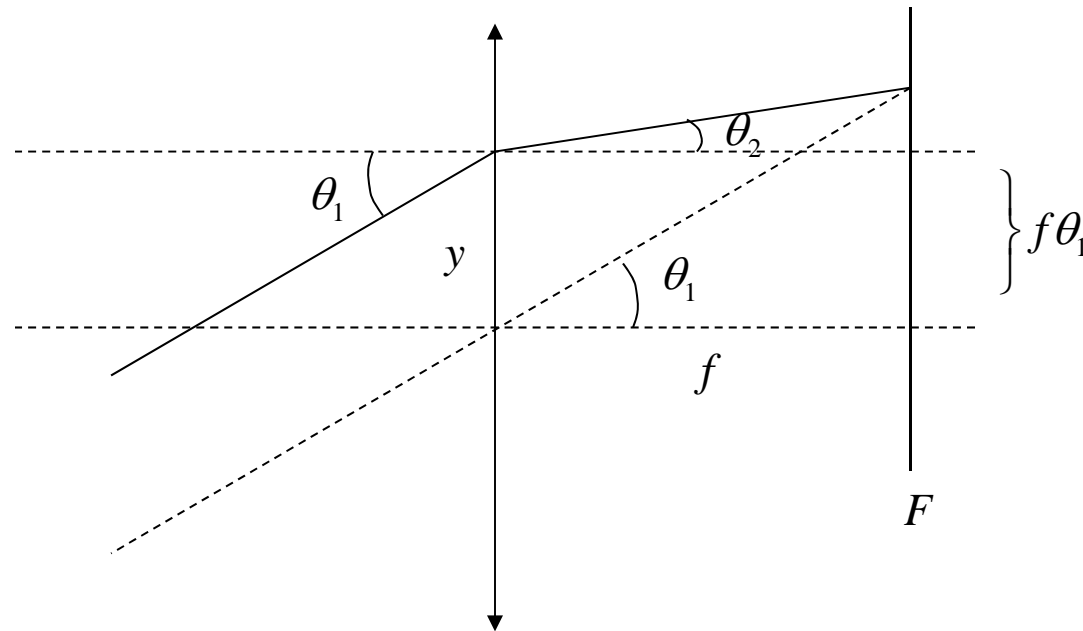


Figure 5-7. Propagation through a lens.

$$\begin{aligned}
 y_2 &= y_1 \\
 \theta_2 &= \frac{f\theta_1 - y_1}{f} \\
 &= \theta_1 - \frac{y_1}{f}
 \end{aligned}
 \tag{5.41}$$

Thus the lens transfer matrix is

$$M_f = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}
 \tag{5.42}$$

These simple rules of transforming  $y$  and  $\theta$  are capable of solving complicated geometrical optics problems, as combining multiple optical components reduces to matrix multiplication. Let us consider the optical arrangement shown in Fig. 8.

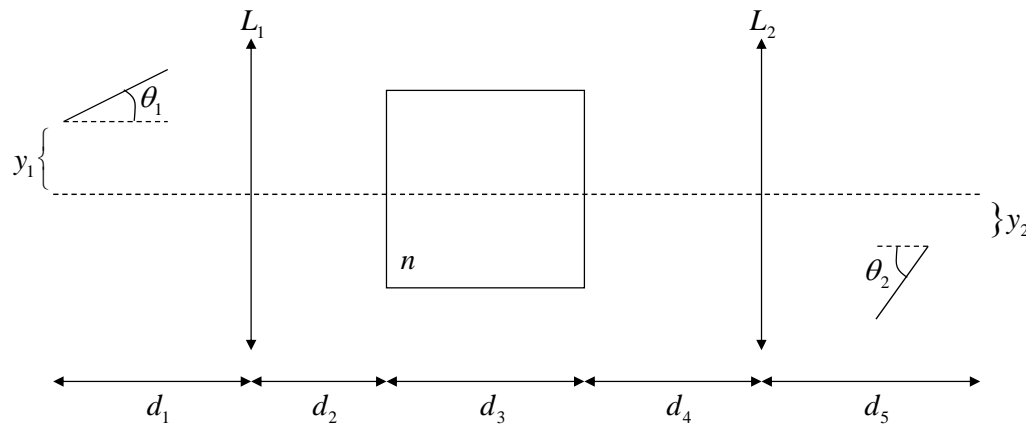


Figure 5-8. Cascading many optical elements.

The ray starts in the configuration  $(y_1, \theta_1)$ , translates a distance  $d_1$ , passes through lens  $L_1$  (focal distance  $f_1$ ), translates a distance  $d_2$ , refracts at the interface with a piece of glass, translates a distance  $d_3$ , refracts back to air, translates a distance  $d_4$ , passes through lens  $L_2$ , and translates a distance  $L_5$ , to end in configuration  $(y_2, \theta_2)$ .

The final configuration can be obtained via matrix multiplication in the proper order,

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = M \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} \tag{5.43}$$

$$M = M_{d_5} \cdot M_{f_2} \cdot M_{d_4} \cdot M_{n1} \cdot M_{d_3} \cdot M_{1n} \cdot M_{d_2} \cdot M_{f_1} \cdot M_{d_1}$$

Note that the order of matrix multiplication is crucial. One way to remember the correct order is to think of it as “chronological,” i.e. the first matrix applied to  $(y_1, \theta_1)$  is the translation matrix  $M_{d_1}$ , followed by lens  $L_1$ , and so on.

The transfer matrices associated with optical systems are sometimes referred to as “ABCD matrices,” where the matrix elements are specifically labeled as  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ .

Now let us see how this matrix formalism applies to Gaussian beams.

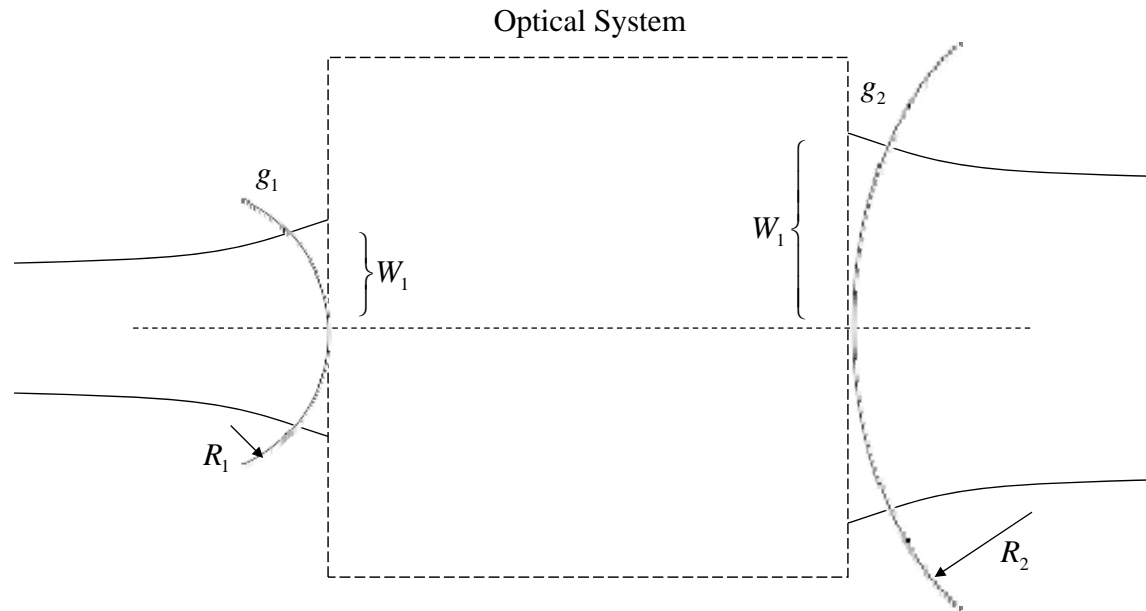


Figure 5-9. Propagation of Gaussian beams through an optical system.

The goal is to find the transformation rules for Gaussian beam parameters, such as the radius of curvature  $R$  and waist  $W$ . Recall that the Gaussian beam is described as (Eq. 26)

$$U(r, z) \propto e^{-\frac{r^2}{c^2(z)}} \quad (5.44)$$

where we ignore the factor  $\frac{1}{c^2(z)}$ , with  $c^2(z) = W_0^2 \left(1 + i \frac{z}{z_0}\right)$ . Thus, the  $z$  dependence of the exponent  $-\frac{r^2}{c^2(z)}$  is carried by the complex quantity,

$$q(z) = z - iz_0. \quad (5.45)$$

In Eq. 45,  $q(z)$  is referred to as the *complex beam parameter* and its transformation rules are of interest here. It can be easily shown that  $g(z)$  can be expressed in terms of both R and W, as follows

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i \frac{\lambda}{\pi W^2(z)} \quad (5.46)$$

Note that a Gaussian beam is fully characterized by the complex beam parameter  $g(z)$ . For an optical system characterized by matrix

$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ , such as shown in Fig. 9,  $g_1$  and  $g_2$  are related by

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}. \quad (5.47)$$

Equation 47 establishes that the transfer matrices can be efficiently used to propagate Gaussian beams through complicated optical systems. Again, the overall matrix that characterizes a cascade of optical elements is obtained by simple matrix multiplication.

*Example 1*

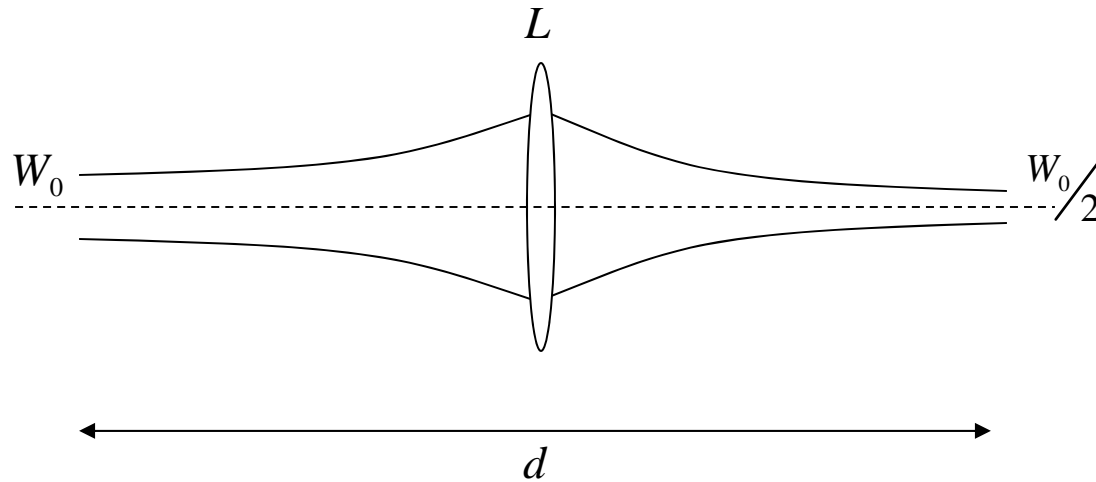


Figure 10. Focusing down a Gaussian beam by a convergent lens.

A Gaussian beam of  $\theta_1 = 1\text{mrad}$  divergence beam is focused down by a lens to half its minimum waist at a distance  $d=1\text{m}$ . What is the focal distance of the lens?

First, we find that the overall matrix of the system is

$$\begin{aligned}
 M &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\
 &= \begin{pmatrix} 1 - \frac{d_1}{f} & d_1 + d_2 - \frac{d_1 d_2}{f} \\ -\frac{1}{f} & 1 - \frac{d_2}{f} \end{pmatrix} \quad (5.48)
 \end{aligned}$$

The radius of curvature is infinite at both planes, such that using Eq. 46, we obtain

$$\begin{aligned}
q_1 &= j \frac{\pi W_0^2}{\lambda} = jz_{0_1} \\
q_2 &= j \frac{\pi W_0^2}{4\lambda} = jz_{0_2}
\end{aligned}
\tag{5.49}$$

Using that  $g_2 = (Ag_1 + B)/(Cg_1 + D)$ , we obtain

$$\begin{aligned}
\frac{z_{0_1} z_{0_2}}{f} &= d_1 + d_2 - \frac{d_1 d_2}{f} \\
z_{0_2} \left(1 - \frac{d_2}{f}\right) &= z_{0_1} \left(1 - \frac{d_1}{f}\right)
\end{aligned}
\tag{5.50}$$

Using the information that  $d_1 + d_2 = 1\text{m}$ , after some manipulations, we obtain  $f = 0.19\text{m}$ .