3. Spatiotemporal field correlations

3.1. Spatiotemporal correlation function. Coherence volume.

- All optical fields in practice fluctuate randomly in both time and space and are subject to a *statistical* description [1]. These fluctuations depend on both the emission process (primary sources) and propagation media (secondary sources).
- *Optical coherence* is a manifestation of the *field statistical similarities* in space and time and coherence theory is the discipline that mathematically describes these similarities [2]. A deterministic field distribution in both time and space is the *monochromatic plane wave*, which is only a mathematical construct, impossible to obtain in practice due to the uncertainty principle.
- The formalism presented below for describing the field correlations is mathematically similar to that used for mechanical fluctuations, for example, in the case of vibrating membranes.

- The analogy between the two different types of fluctuations and their mathematical description in terms of spatiotemporal correlations has been recently emphasized [3].
- A starting point in understanding the physical meaning of a *statistical optical field* is the question: what is the *effective (average) temporal* sinusoid, $\left\langle e^{-i\omega t}\right\rangle_{\omega}$, for a broadband field? What is the *average spatial* sinusoid, $\left\langle e^{i\mathbf{k}\cdot\mathbf{r}}\right\rangle_{\mathbf{k}}$.
- A monochromatic plane wave is described by $e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})}$. These two averages can be performed using the *probability densities* associated with the temporal and spatial frequencies, *S(ω)* and *P(**k**)*, which are normalized to satisfy $\int S(\omega)d\omega=1$, $\int P(\mathbf{k})d^3\mathbf{k}=1$.
- Thus, *S(ω)dω* is the probability of having frequency component ω in our field, or the fraction of the total power contained in the vicinity of frequency *ω*.

- Similarly, $P(\boldsymbol{k})d^3\boldsymbol{k}$ is the probability of having component $\boldsymbol{k}$ in the field, or the fraction of the total power contained around spatial frequency $\boldsymbol{k}$**.** Up to a normalization factor, *S and P are the temporal and spatial power spectra* associated with the fields. The two "effective sinusoids" can be expressed as *ensemble averages*, using $S(\omega)$ and $P(\boldsymbol{k})$ as weighting functions,

$$\begin{aligned}\left\langle e^{-i\omega t}\right\rangle_{\omega} &= \int S(\omega)e^{-i\omega t}d\omega \\ &= \Gamma(t)\end{aligned} \qquad (1a)$$

$$\begin{aligned}\left\langle e^{i\mathbf{k}\cdot\mathbf{r}}\right\rangle_{\mathbf{k}} &= \int P(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{k} \\ &= W(\mathbf{r})\end{aligned} \qquad (1b)$$

- Equations 1a-b establish that the average *temporal sinusoid* for a broadband field equals its temporal autocorrelation, Γ. The average *spatial sinusoid* for an inhomogeneous field equals its spatial autocorrelation, denoted by *W*.

- Besides describing the statistical properties of optical fields, coherence theory can make predictions of experimental relevance. The general problem can be formulated as follows (Fig. 1):

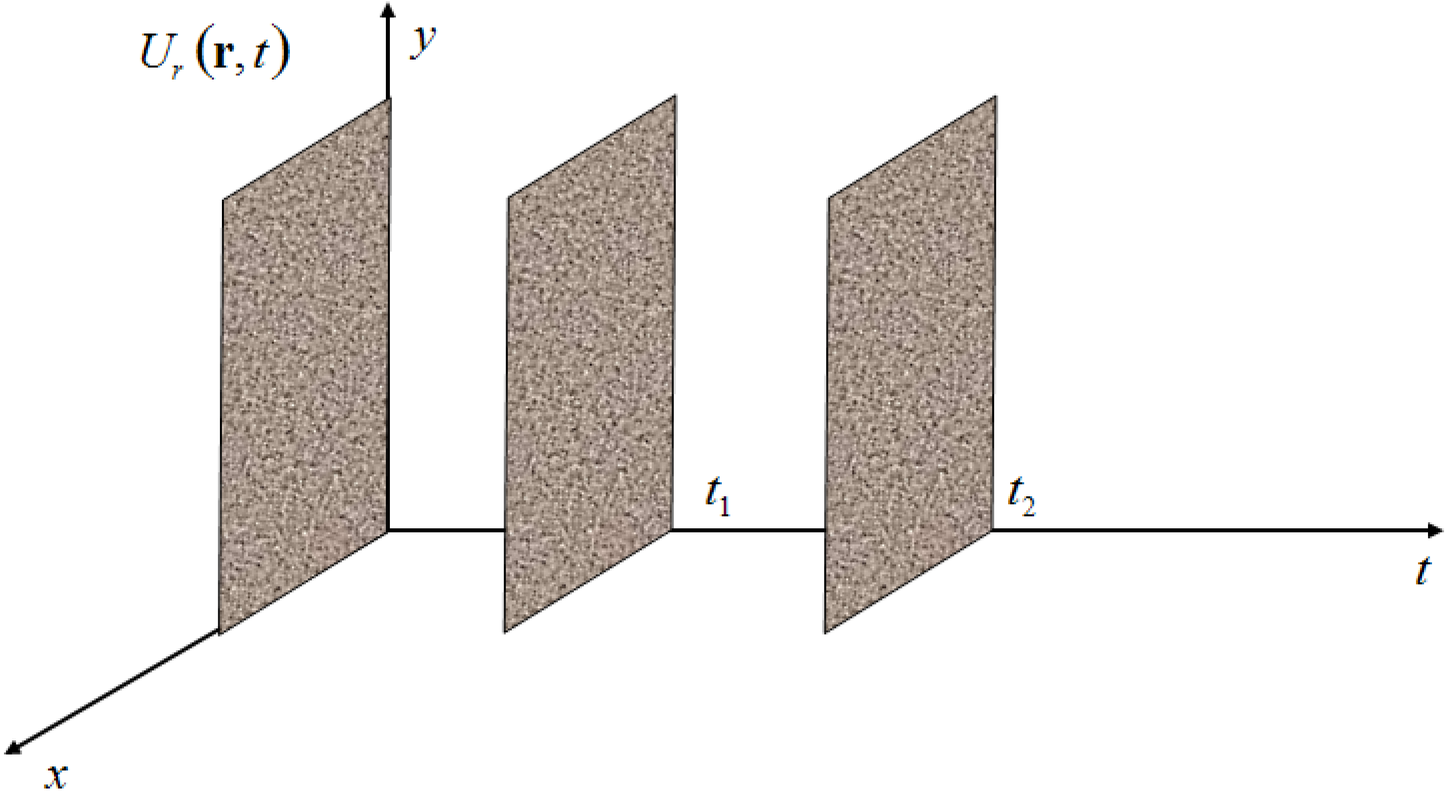

Figure 1. Spatio-temporal distribution of a real optical field

- Given the optical field distribution $U(\mathbf{r},t)$ that varies randomly in space and time, over what *spatiotemporal domain* does the field preserve significant correlations? This translates into: combining the field $U(\mathbf{r},t)$ with a replica of itself shifted in both time and space, $U(\mathbf{r}+\boldsymbol{\rho},\ t+\tau)$, *on average*, how large can $\boldsymbol{\rho}$ and τ be and still observe "significant" interference?
- We expect that monochromatic fields exhibit infinitely broad temporal correlations, plane waves are expected to manifest broad spatial correlations. Regardless of how much we shift a monochromatic field in time or a plane wave in space, they remain perfectly correlated with their unshifted replicas. It is difficult to picture temporal correlations decaying over timescales that are shorter than an optical period and spatial correlations that decay over spatial scales smaller than the optical wavelength. In the following we provide a quantitative description of the spatiotemporal correlations.

- The statistical behavior of optical fields can be mathematically captured generally via a *spatiotemporal correlation function*

$$\Lambda\left(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2\right)=\left\langle U\left(\mathbf{r}_1,t_1\right)\cdot U^*\left(\mathbf{r}_2,t_2\right)\right\rangle_{\mathbf{r},t} \qquad 2$$

- The average is performed *temporally and spatially*, indicated by the subscripts **r** and *t*. Because common detector arrays capture the spatial intensity distributions in 2D only, we will restrict the discussion to $\mathbf{r}=(x,y)$, without losing generality. These averages are defined in the usual sense as

$$\begin{aligned}\left\langle U\left(\mathbf{r}_1,t_1\right)\cdot U^*\left(\mathbf{r}_2,t_2\right)\right\rangle_t &= \lim_{T\to\infty}\frac{1}{T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} U\left(\mathbf{r}_1,t_1\right)\cdot U^*\left(\mathbf{r}_2,t_2\right)dt_1dt_2 \\ \left\langle U\left(\mathbf{r}_1,t_1\right)\cdot U^*\left(\mathbf{r}_2,t_2\right)\right\rangle_{\mathbf{r}} &= \lim_{A\to\infty}\frac{1}{A^2}\int_A\int_A U\left(\mathbf{r}_1,t_1\right)\cdot U^*\left(\mathbf{r}_2,t_2\right)d^2\mathbf{r}_1d^2\mathbf{r}_2\end{aligned} \qquad 3$$

- Often we deal with fields that are both *stationary* (in time) and *statistically homogeneous* (in space).

- If stationary, the statistical properties of the field (e.g. the average, higher order moments) do not depend on the origin of time. Similarly, for statistically homogeneous fields, their properties do not depend on the origin of space. Wide-sense stationarity is less restrictive and defines a random process with only it's first and second moments independent of the choice of origin. For the discussion here, the fields are assumed to be stationary at least in the wide-sense.
- Under these circumstances, the dimensionality of the spatiotemporal correlation function Λ decreases by half,

$$\begin{aligned} \Lambda(t_1, t_2) &= \Lambda(t_2 - t_1) \\ \Lambda(\mathbf{r}_1, \mathbf{r}_2) &= \Lambda(\mathbf{r}_2 - \mathbf{r}_1) \end{aligned} \qquad 4$$

- The spatiotemporal correlation function becomes

$$\Lambda(\boldsymbol{\rho}, \tau) = \left\langle U(\mathbf{r}, t) \cdot U^*(\mathbf{r} + \boldsymbol{\rho}, t + \tau) \right\rangle_{\mathbf{r}, t} \qquad 5$$

- $\Lambda(\mathbf{0},0)=\left\langle U(\mathbf{r},t)\cdot U^*(\mathbf{r},t)\right\rangle_{\mathbf{r},t}$ represents the *spatially averaged irradiance* of the field, which is, of course, a real quantity. In general $\Lambda(\boldsymbol{\rho},\tau)$ is complex. Define a normalized version of Λ, referred to as the *spatiotemporal complex degree of correlation*

$$\alpha(\boldsymbol{\rho},t)=\frac{\Lambda(\boldsymbol{\rho},\tau)}{\Lambda(\mathbf{0},0)}. \qquad 6$$

- For stationary fields $|\Lambda|$ attains its maximum at $\mathbf{r}=0,\quad t=0$, thus

$$0<\left|\alpha(\rho,\tau)\right|<1 \qquad 7$$

- Define an area $A_C\propto\rho_C{}^2$ and length $l_c=c\tau_C$, over which $\left|\alpha(\rho_C,\tau_C)\right|$ maintains a significant value, say $|\alpha|>1/2$, which defines a *coherence volume*

$$V_C=A_C\cdot l_C. \qquad 8$$

- This *coherence volume* determines the maximum domain size over which the fields can be considered correlated. In general an extended source, such as an incandescent filament, may have spectral properties that vary from point to point. It is convenient to discuss spatial correlations at each frequency ω, as described below.

3.2. Spatial correlations of monochromatic light

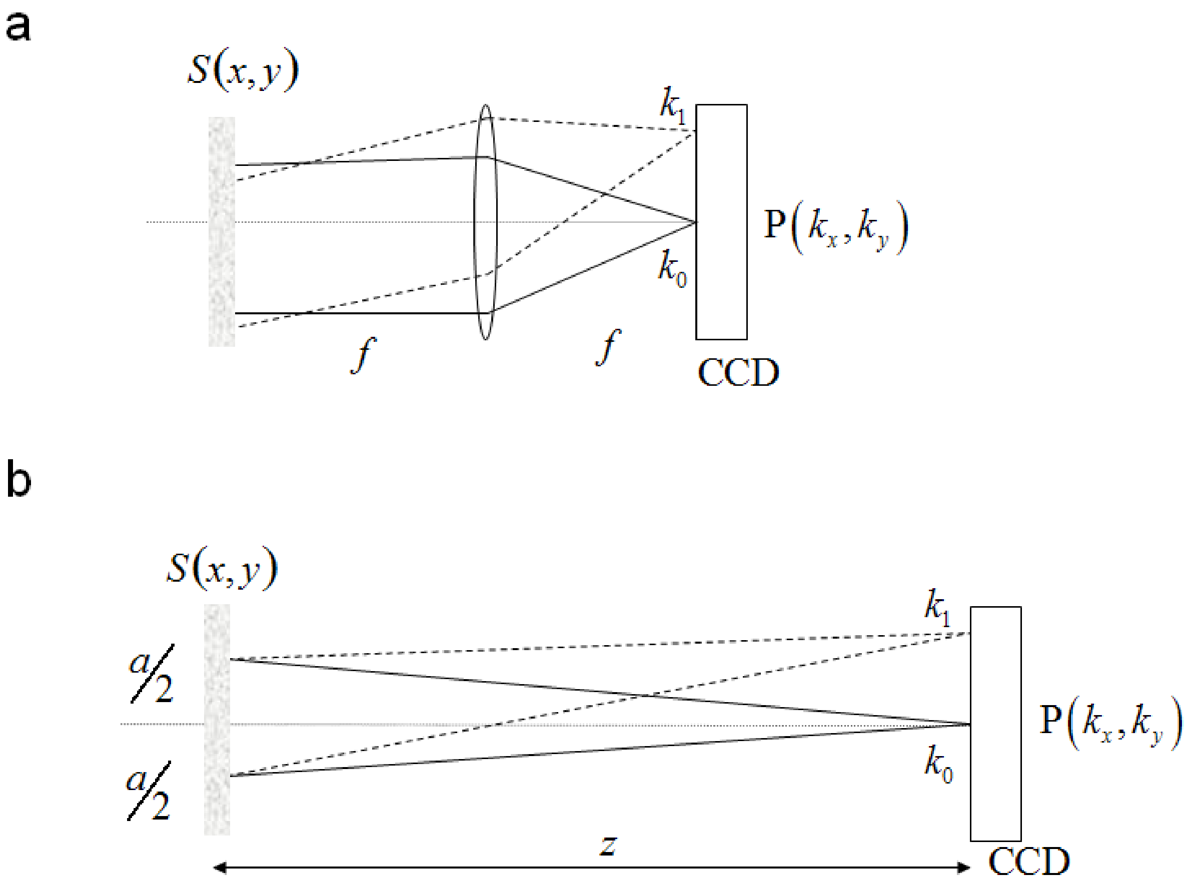

Figure 3. Measuring the spatial power spectrum of the field from source S via a lens (a) and Fraunhofer propagation in free space (b).

- The spatial correlation function $W\left(\boldsymbol{\rho},\omega\right)$ can also be experimentally determined from measurements of the spatial power spectrum, as shown in Fig. 3.
- Both the far field propagation in free space and propagation through a lens can generate the Fourier transform of the source field, as illustrated in Fig. 3,

$$\tilde{U}\left(\mathbf{k},\omega\right)=\int U\left(\mathbf{r},\omega\right)\cdot e^{-i\mathbf{kr}}d^{2}\mathbf{r}, \tag{16}$$

- The CCD is sensitive to power and detects the spatial power spectrum, $P\left(\mathbf{k},\omega\right)=\left|\tilde{U}\left(\mathbf{k},\omega\right)\right|^{2}$.
- In Eq. 16, the frequency component $\mathbf{k}=(k_x,k_y)$ depends either on the focal distance, for the lens transformation (Fig. 3a), or on the propagation distance z, for the Fraunhofer propagation (Fig. 3b),

$$\mathbf{k} = \frac{2\pi}{\lambda f}\left(x', y'\right)$$
$$. \qquad 17$$
$$\mathbf{k} = \frac{2\pi}{\lambda z}\left(x', y'\right)$$

- In the Fraunhofer regime, the ratios *x/f* and *x/z* describe the diffraction angle; therefore sometimes $P\left(\mathbf{k},\omega\right)$ is called *angular power spectrum.*
- For extended sources that are far away from the detection plane, as in Fig. 3b, the size of the source may have a significant effect on the Fourier transform in Eq. 16. This effect becomes obvious if we replace the source field *U* with its spatially truncated version, $\underline{U}$, to indicate the finite size of the source

$$\underline{U}\left(\mathbf{r},\omega\right) = U\left(\mathbf{r},\omega\right)\cdot\Pi\left(\frac{\mathbf{r}}{a}\right), \qquad 18$$

- Π is the *2D rectangular function*, a square of side a. The far field becomes

$$\tilde{\underline{U}}\left(\mathbf{k},\omega\right) = a^2\tilde{U}\left(\mathbf{k},\omega\right) *_{k_x k_y} \operatorname{sinc}\left(a\mathbf{k}\right), \qquad 19$$

- * denotes convolution and *sinc* is sin(x)/x.
- The field across detection plane *(x', y')*, $\underline{\tilde{U}}\left(\mathbf{k},\omega\right)$, is *smooth* over scales given by the width of the *sinc* function.
- This *smoothness* indicates that the field is spatially correlated over this spatial scale. Along x', this correlation distance, x_c', is obtained by writing explicitly the spatial frequency argument of the *sinc* function,

$$\begin{aligned}\frac{2\pi}{a} &= k_x \\ &= \frac{2\pi}{\lambda z}\cdot x_c'\end{aligned} \qquad (20)$$

- We can conclude that the correlation area of the field generated by the source in the far zone is of the order of

$$\begin{aligned} A_C &= x_c{}^2 = \\ &= \frac{\lambda^2}{\Delta\Omega}. \end{aligned} \qquad 21$$

- Ω is the solid angle subtended by the source.
- This relationship allowed Michelson to measure interferometricaly the angle subtended by stars.
- For example, the Sun subtends an angle $\theta \simeq 10\,\text{mrad}$, i.e. $\Omega \simeq 10^{-4}\,\text{srad}$. Thus, for the green radiation that is the mean of the visible spectrum, $\lambda = 550\text{nm}$, the coherence area at the surface of the Earth is of the order of $A_C{}^{\text{Sun}} = 50 \cdot 50\,\mu\text{m}^2$. Measuring this area over which the sun light shows correlations (or generates fringes) provides information about its angular size.
- For angularly smaller sources, far field spatial coherence is correspondingly higher. **This is the essence of the Van Cittert-Zernike theorem, which states**

that the field generated by spatially incoherent sources gains coherence upon propagation. This is the result of free-space propagation acting as a *spatial low-pass filter [4]*.

- Zernike employed the spatial filtering concept to develop *phase contrast microscopy* [5, 6]. It had been known since Abbe that an image can be described as an interference phenomenon [7]. Image formation is the result of simultaneous interference processes that take place at each point in the image.

3.3. Temporal correlations of plane waves

- We now investigate the temporal correlations of fields at a particular spatial frequency **k** (or a certain direction of propagation). Taking the *spatial* Fourier transform of Λ in Eq. 2, we obtain the *temporal correlation function*

$$\begin{aligned}\Gamma(\mathbf{k},\tau) &= \iint \Lambda(\tilde{\rho},\tau)\cdot e^{-i\mathbf{k}\rho} d^2\mathbf{r} \\ &= \left\langle U(\mathbf{k},t)\cdot U^*(\mathbf{k},t+\tau)\right\rangle_t\end{aligned} \tag{22}$$

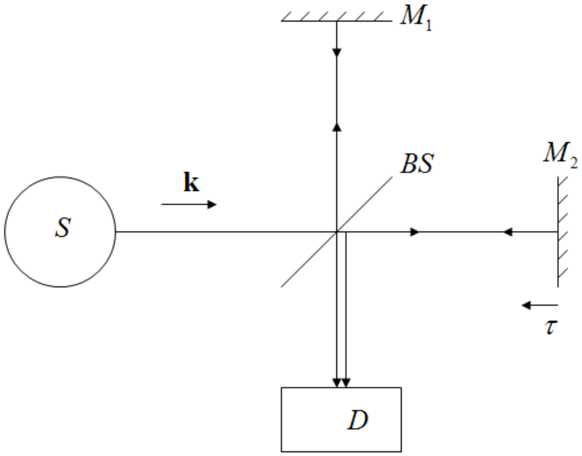

Figure 5. Michelson interferometry.

- The autocorrelation function Γ is relevant in interferometric experiments of the type illustrated in Fig. 5. In a Michelson interferometer, a plane wave from the source is split in two by the beam splitter and subsequently recombined via reflections on mirrors M_1,and M_2. The intensity at the detector has the form (we assume 50/50 beam splitter)

$$\begin{aligned} I(\mathbf{k},\tau) &= \left\langle \left| U(\mathbf{k},t) + U^*(\mathbf{k},t+\tau) \right|^2 \right\rangle_t \\ &= 2I(\mathbf{k}) + 2\,\mathrm{Re}\left\langle U(\mathbf{k},t)\cdot U^*(\mathbf{k},t+\tau) \right\rangle \end{aligned} \qquad 23$$

- The real part of $\Gamma(\mathbf{k},\tau)$ is obtained by varying the time delay between the two fields. This delay can be controlled by translating one of the mirrors. The *complex degree of temporal correlation* at spatial frequency **k** is defined as

$$\gamma(\mathbf{k},\tau) = \frac{\Gamma(\mathbf{k},\tau)}{\left|\Gamma(\mathbf{k},0)\right|} \qquad 24$$

- $\Gamma(\mathbf{k},0)$ represents the intensity of the field, i.e.

$$\begin{aligned}\Gamma(\mathbf{k},0) &= \left\langle U(\mathbf{k},t)\cdot U^*(\mathbf{k},t)\right\rangle_t \\ &= I(\mathbf{k})\end{aligned} \qquad 25$$

- The complex degree of temporal correlation has the similar property with its spatial counterpart β, i.e.

$$0<\left|\gamma(\mathbf{k},\tau)\right|<1 \qquad 26$$

- The *coherence time* is defined as the maximum time delay between the fields for which $|\gamma|$ maintains a significant value, say ½.

- If we cross-correlate temporally two plane waves of different wave vectors (directions of propagation), the result vanishes unless $\mathbf{k}_1=\mathbf{k}_2$,

$$\begin{aligned}\Gamma(\mathbf{k_1},\mathbf{k_2},\tau) &= \left\langle U_1(\mathbf{k}_1,t)\cdot U_2^{\,*}(\mathbf{k}_2,t+\tau)\right\rangle_t \\ &= \Gamma(\mathbf{k_1},\mathbf{k}_1,\tau)\delta(\mathbf{k}_2-\mathbf{k}_1)\end{aligned} \qquad 27$$

- At each moment t, the two plane waves generate fringes parallel to $\mathbf{k}_2 - \mathbf{k}_1$. If the detector (e.g. a CCD) averages the signal over scales larger than the fringe period, the temporal correlation information is lost.
- As τ changes, the fringes "run" across the plane such that the contrast averages to 0. For this reason, for example, the two beams in a typical Michelson interferometer are carefully aligned to be parallel.
- The temporal correlation Γ is the Fourier transform of the power spectrum,

$$\begin{aligned} \Gamma(\mathbf{k},\tau) &= \int_{-\infty}^{\infty} S(\mathbf{k},\omega)\cdot e^{-i\omega\tau}d\omega \\ S(\mathbf{k},\omega) &= \int_{-\infty}^{\infty} \Gamma(\mathbf{k},\tau)\cdot e^{i\omega\tau}d\omega \end{aligned} \tag{28}$$

- Γ can be determined via spectroscopic measurements, as exemplified in Fig. 6.

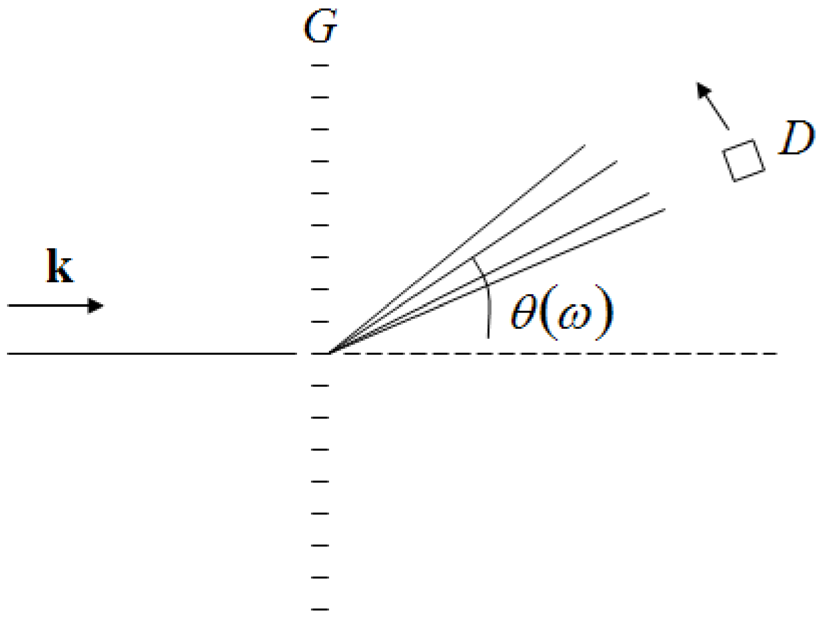

Figure 6. Spectroscopic measurement using a grating: G grating, D detector, diffraction angle. The dashed line indicates the undiffracted order (zeroth order)

- By using a grating (a prism, or any other dispersive element), we can "disperse" different colors at different angles, such that a rotating detector can measure $S(\omega)$ directly.
- To estimate the coherence time for a broad band field, let us assume a Gaussian spectrum centered at frequency ω_0, and having the r.m.s. width $\Delta\omega$,

$$S(\omega) = S_0 \cdot e^{-\left(\frac{\omega-\omega_0}{\sqrt{2}\Delta\omega}\right)^2}, \qquad 29$$

- S_0 is a constant.
- The autocorrelation function is also a Gaussian, modulated by a sinusoidal function, as a result of the Fourier shift theorem

$$\Gamma(\tau) = \Gamma_0 \cdot e^{-\left(\frac{\Delta\omega\tau}{\sqrt{2}}\right)^2} \cdot e^{i\omega_0\tau}$$

30If we define the width of Γ as the coherence time, we obtain

$$\tau_C \propto \frac{1}{\Delta\omega}, \qquad 31$$

- and the coherence length

$$\begin{aligned} l_C &= c\tau_C \\ &\propto \frac{\lambda^2}{\Delta\lambda} \end{aligned} \qquad 32$$

- The coherence length depends on the spectral bandwidth in an analog fashion to the coherence area dependence on solid angle (Eq. 21). This is not surprising as both types of correlations depend on their respective frequency bandwidth.

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