$U(\hat{r},t) = A(\hat{r},t)\, e^{i\phi(\hat{r},t)}$ $\quad e^{-i(\omega t - \bar{k}\bar{r})}$

deterministic

4. STATISTICAL PROPERTIES OF OPTICAL FIELDS

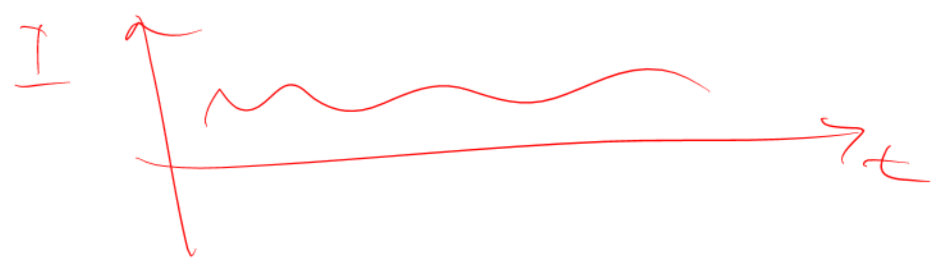
All optical fields encountered in practice fluctuate randomly in both time and space and are, therefore, subject to a *statistical* description. These field fluctuations depend on both the emission process (primary sources) and propagation media (secondary sources). Thus, *optical coherence* is a manifestation of the *field statistical similarities* and coherence theory is the discipline that mathematically describes these similarities.

How can we start to think of *statistically averaged* fields? Temporally, the question becomes: what is the *effective, i.e. average temporal* sinusoid, i.e. $\left\langle e^{-i\omega t}\right\rangle_{\omega}$, for a broadband field? Similarly, what is the *average spatial* sinusoid, i.e. $\left\langle e^{i\mathbf{k}\cdot\mathbf{r}}\right\rangle_{\mathbf{k}}$. These averages can be performed using the *probability densities* associated with the temporal and spatial frequencies, *S(ω)* and *P(**k**)*, which satisfy $\int S(\omega)d\omega = 1$, $\int P(\mathbf{k})d^3\mathbf{k} = 1$. Thus, *S(ω)dω* is the probability of having frequency component ω in our field, or the fraction of the total power contained in the vicinity of frequency *ω*. Similarly, $P(\boldsymbol{k})d^3\boldsymbol{k}$ is the probability of having component ***k*** in the field, or the fraction of the total power contained around spatial frequency ***k*.** Up to a normalization factor, S and P are the temporal and power spectra associated with the fields. Thus, the two "effective sinusoids" can be expressed as *ensemble averages*, using *S(ω)* and *P(**k**)* as weighting functions,

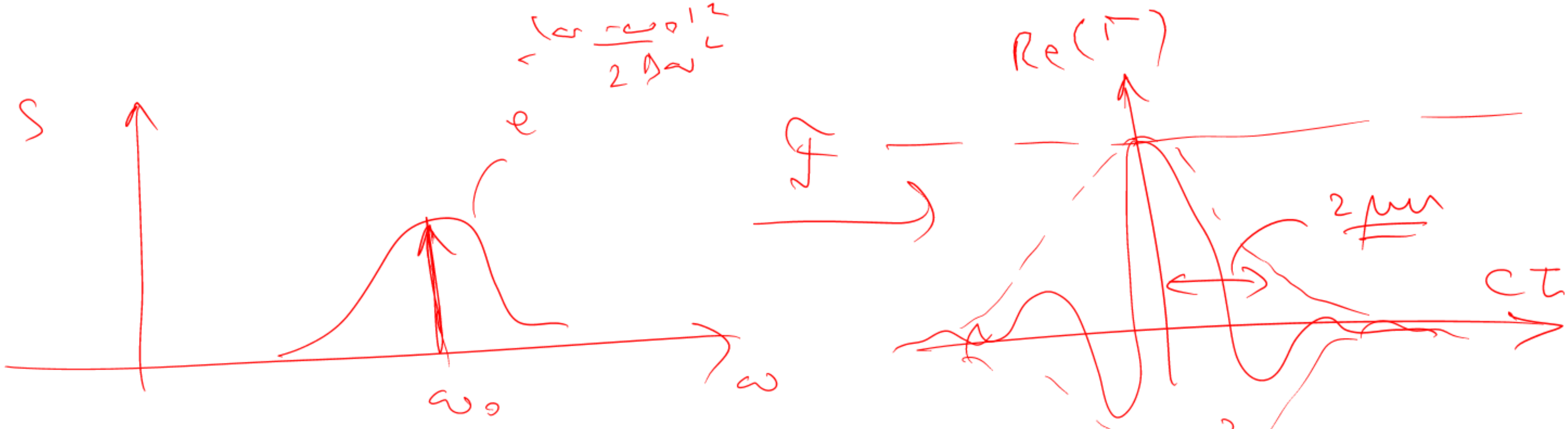
$$\left\langle e^{-i\omega t}\right\rangle_{\omega} = \int S(\omega)e^{-i\omega t}d\omega = \Gamma(t) \quad \text{(Ia)}$$

$$\left\langle e^{i\mathbf{k}\cdot\mathbf{r}}\right\rangle_{\mathbf{k}} = \int P(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{k} = W(\mathbf{r}) \quad \text{(Ib)}$$

Equations Ia-b establish that the average temporal (spatial) "sinusoid" for a broadband (inhomogeneous) field equals its temporal (spatial autocorrelation), denoted by *Γ (W).* In the following, we will study these spatiotemporal correlations and determine their effects in experiments involving interference.



Ensemble average



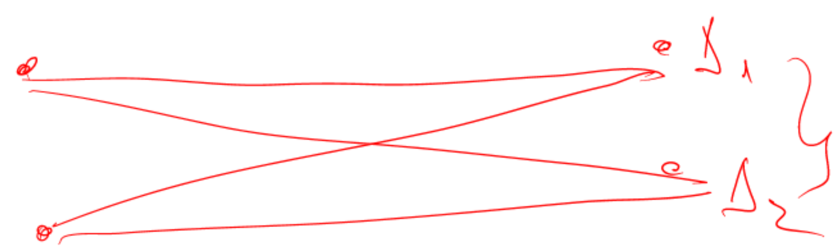
$$e^{-\frac{(\omega-\omega_0)^2}{2\Delta\omega^2}} \longrightarrow \underbrace{e^{-\Delta\omega^2\tau^2}}_{\text{env.}} \cdot \underbrace{e^{i\omega_0\tau}}_{\text{carrier}} = \underline{\underline{\Gamma(\tau)}}$$

FT

Coherence time

τ_c

$\underline{l_c = c\,\tau_c}$



van Cittert - Zernike theorem.

ω → spatial coherence

$\bar{k}$ || → temporal coh.

4.1 Complex representation of real fields

It is very convenient to describe a given real signal via an associated complex quantity. This way, many nuisances due to trigonometry are avoided. The key is to realize that a real signal $U(t)$ has a Fourier transform $\tilde{U}(\omega)$ that is Hermitian,

$e^{-i\omega t + \bar{k}\cdot\bar{r}}$

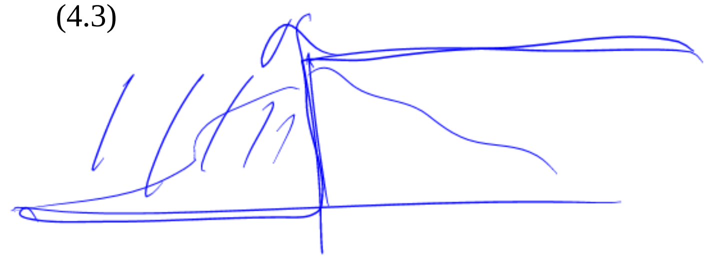
$$\begin{cases} U_r(t) = \int_{-\infty}^{\infty} \tilde{U}(\omega)\cdot e^{-i\omega t}\,d\omega, \\ \boxed{U(-\omega) = U^*(\omega).} \end{cases} \tag{4.1}$$

Thus, the information contained in the negative frequencies is identical with that contained in positive frequencies. Therefore, suppressing the negative frequencies in Eq. 1a does not generate any loss of information. With this operation, we define a new function, called the *complex analytic* signal associated with U_r,

$$\boxed{U(t) = 2\int_{0}^{\infty} \tilde{U}(\omega)\cdot e^{i\omega t}\,d\omega,} \tag{4.2}$$

where the factor of 2 will become clear shortly. Functions $U(t)$ is now complex, because its Fourier transform misses the negative frequencies and is, thus, non-Hermitian. In order to find an explicit relationship between $U(t)$ and $U_r(t)$, let us express Eq. 2 as

$$\begin{aligned} U(t) &= 2\int_{-\infty}^{\infty} \underbrace{\tilde{U}(\omega)\cdot H(\omega)}\cdot e^{-i\omega t}\,d\cancel{t}\omega \\ H(\omega) &= \begin{vmatrix} 1, & \omega > 0 \\ 1/2, & \omega = 0, \\ 0, & \omega < 0 \end{vmatrix} \end{aligned} \tag{4.3}$$



Where $H(\omega)$ is known as the Heaviside function. The Fourier transform of the product in Eq. 3a is the convolution of the two Fourier transforms. The inverse Fourier transforms of $\tilde{U}(\omega)$ and $H(\omega)$ are given by

$$\begin{aligned} &\tilde{U}(\omega) \rightarrow U_r(t) \\ &\boxed{H(\omega) \rightarrow \frac{1}{2}\delta(t) - \frac{i}{2\pi t}.} \end{aligned} \qquad (4.4)$$

V

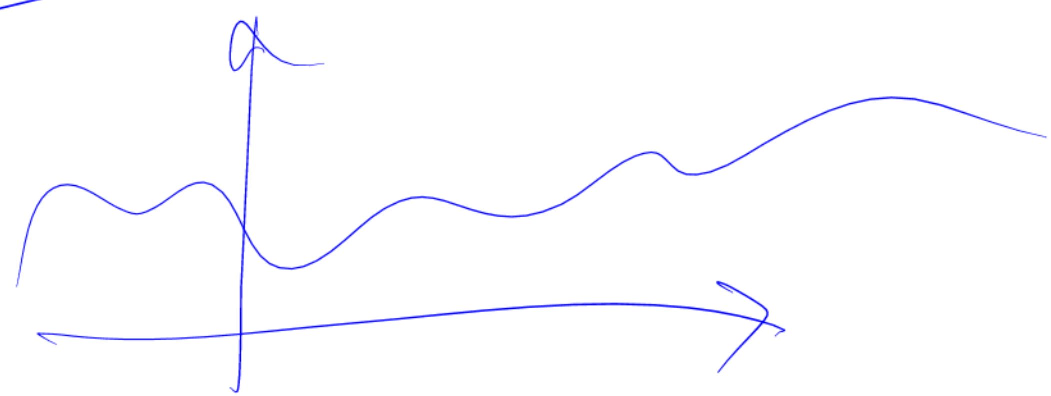
As a result, Eq. 3a can be rewritten via the convolution theorem as

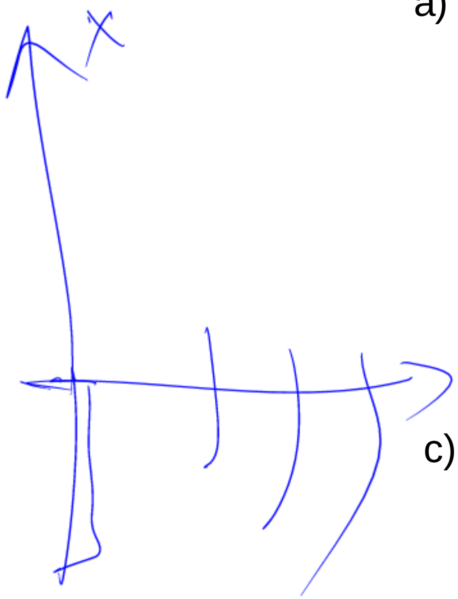
$$\boxed{U(t) = U_r(t) - \frac{i}{\pi} P \int \frac{U_r(t')}{t - t'} dt'} \qquad (4.5)$$

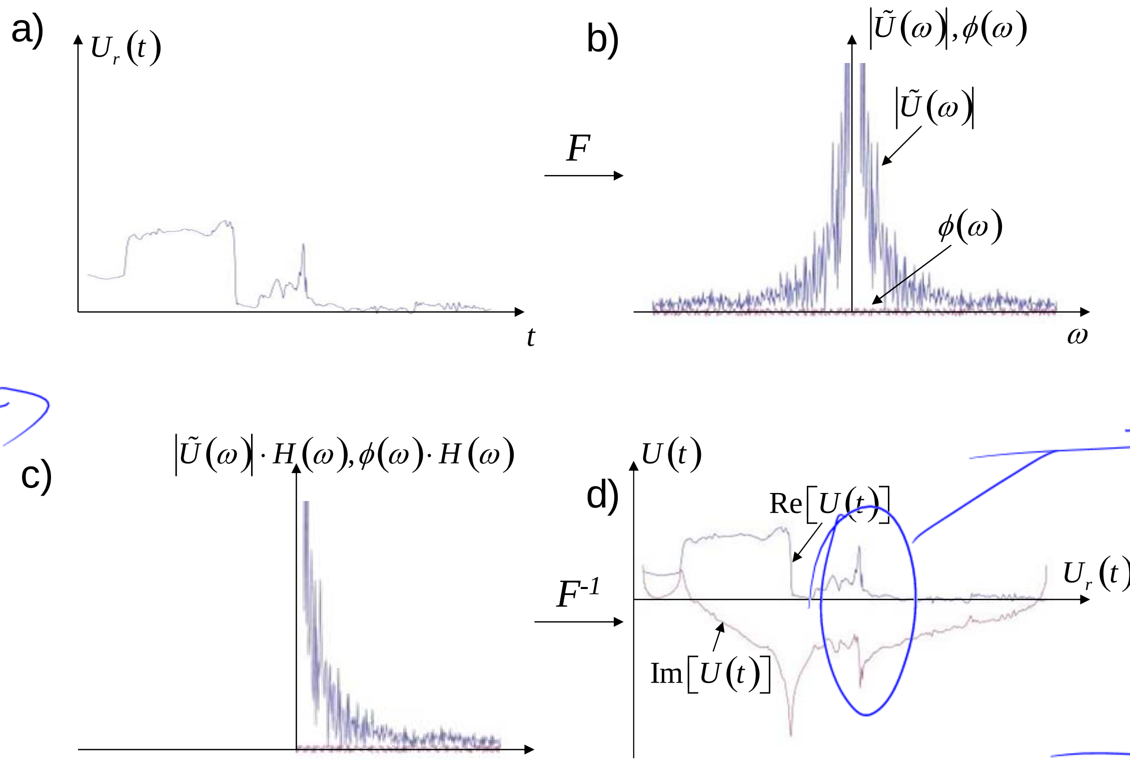
$\in \mathbb{C}$

Figure 1 illustrates the operations involved in computing the complex analytic signal associated with a real field.

Hilbert transf'







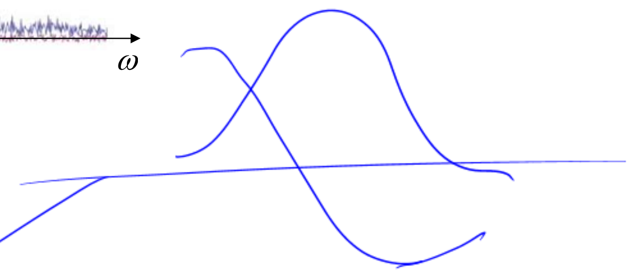


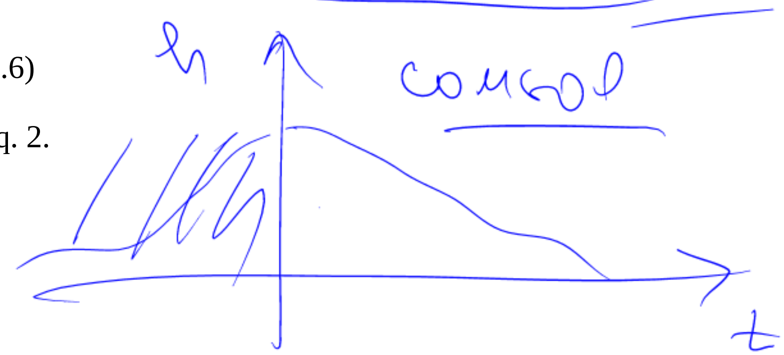
Figure 4-1. From a real function $U_r(t)$ to its analytic signal $U(t)$.

$u(\omega) = u'(\omega) + i u''(\omega)$

Equation 5 reveals very important properties of the *complex analytic signal* U, as follows.

i) The real part of the complex analytic signal recovers the original (real) signal,

$$\mathrm{Re}[U(t)] = U_r(t) \tag{4.6}$$



Note that this relationship holds without scaling factors due to the factor of 2 used in Eq. 2.

$X(\omega) = x' + i x''$

ii) $\mathrm{Re}\left[U(t)\right]$ and $\mathrm{Im}\left[U(t)\right]$ are related via

$$\begin{aligned} U_i(t) &= \mathrm{Im}\left[U(t)\right] \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{U_r(t')}{t-t'}dt' \end{aligned} \tag{4.7}$$

The (principal value) integral in Eq. 7 is known as a *Hilbert transform*. Mathematically, the Hilbert transform of a function $f(x)$ is equivalent to convolving it with function $1/x$. This relationship between the real and imaginary parts of U should come as no surprise, because the imaginary part was generated from simply suppressing the negative frequencies of U_r, i.e. U_i does not contain any new information that is not already in U_r.

iii) The time-averaged modulus squared of the complex analytic signal is proportional to the field *irradiance*,

$$\begin{aligned} \left\langle \left|U(t)\right|^2 \right\rangle_t &= \left\langle U_r^{\,2}(t) + U_i^{\,2}(t) \right\rangle_t \\ &= 2\left\langle U_r^{\,2}(t)\right\rangle_t . \end{aligned} \tag{4.8}$$

With these three properties, we can safely denote a real field by its complex analytic signal, which greatly simplifies the calculations. As a common example, the interference between two fields is computed much faster in the complex representation. Thus in the real field representation, we have (for $a_1, a_2 \in \mathbb{R}$)

$$\begin{aligned} I &= \left\langle \left(a_1\cos(\omega t+\varphi_1) + a_2\cos(\omega t+\varphi_2)\right)^2 \right\rangle_t \\ &= \frac{a_1^{\,2}}{2} + \frac{a_2^{\,2}}{2} + \left\langle 2a_1a_2\cdot\cos(\omega t+\varphi_1)\cdot\cos(\omega t+\varphi_2)\right\rangle_t \\ &= \frac{a_1^{\,2}}{2} + \frac{a_2^{\,2}}{2} + 2a_1a_2\left[\frac{\cos(\varphi_2-\varphi_1)}{2} + \cancel{\frac{\cos(2\omega t+\varphi_1+\varphi_2)}{2}}\right] \end{aligned} \tag{4.9}$$

Typically, the fast term, i.e. containing $2\omega t$ as argument, is ignored, such that the final intensity of the sum field reads

$$I = \frac{a_1^{\ 2}}{2} + \frac{a_2^{\ 2}}{2} + a_1 a_2 \cdot \cos\left(\varphi_2 - \varphi_1\right) \tag{4.10}$$

Now, let us calculate the same quantity via the complex analytic signals (Eq. 8)

$$\begin{aligned} I &= \frac{1}{2}\left\langle \left|U\left(t\right)\right|^2 \right\rangle_t \\ &= \frac{1}{2}\left\langle \left|a_1 \cdot e^{i\varphi_1} + a_2 \cdot e^{i\varphi_2}\right|^2 \right\rangle_t \\ &= \frac{1}{2} a_1^{\ 2} + \frac{1}{2} a_2^{\ 2} + a_1 a_2 \cos\left(\varphi_2 - \varphi_1\right) \end{aligned} \tag{4.11}$$

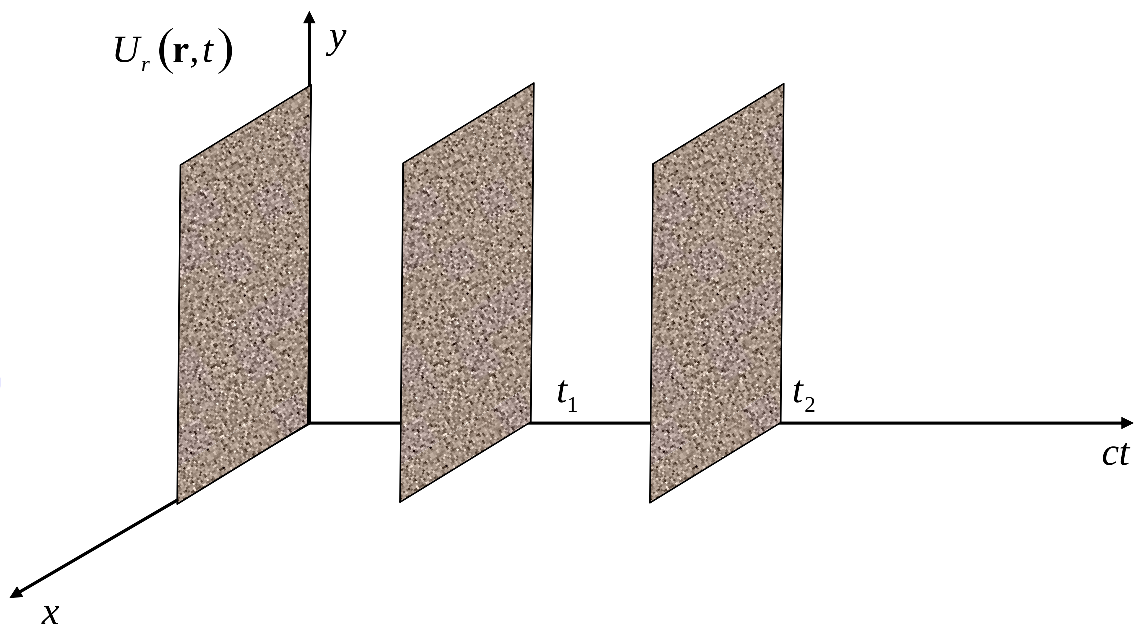
Thus, the complex representation is far more efficient in describing field superposition, especially when a large number of field components is involved.

4.2 Spatiotemporal correlation function. Coherence volume.

Besides the basic science interest in describing the statistical properties of optical fields, coherence theory can make predictions of experimental relevance. The general problem can be formulated like this (Fig. 2): given a certain distribution of optical field $U(\mathbf{r},t)$ that varies randomly in space and time, over what spatiotemporal domain does the field preserve significant correlations? Experimentally, this question translates into: interfering the field $U(\mathbf{r},t)$ with a replica of itself shifted in both time and space, $U(\mathbf{r}+\rho,t+\tau)$, *on average*, how large can $\Delta\mathbf{r}$ and Δt be and still obtain significant interference fringes?

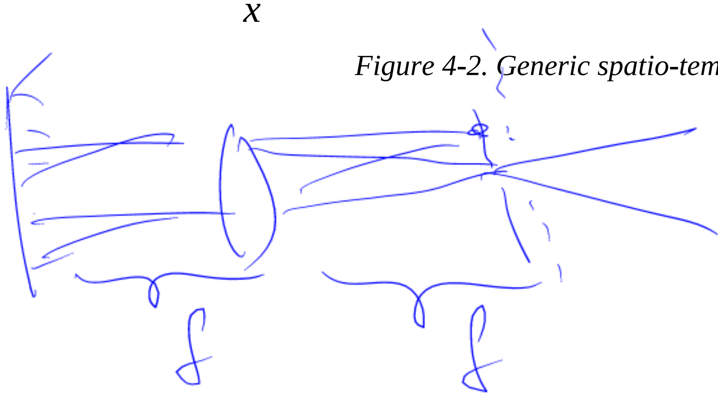
$\tau_c \propto \frac{1}{\Delta\omega}$

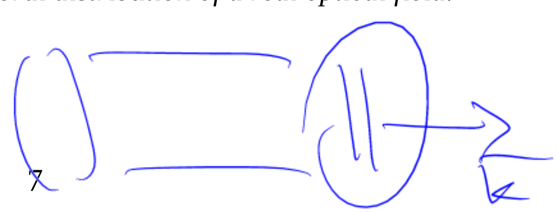
$\boxed{l_c = c\tau_c}$



$c\tau_c = 2\,\mu m$

Figure 4-2. Generic spatio-temporal distribution of a real optical field.





Intuitively, we expect that monochromatic fields will exhibit broad temporal correlations, while plane waves will show broad spatial correlations. On the other hand, it is difficult to picture temporal correlations that decay over timescales that are shorter than an optical period and spatial correlations that decay over smaller spatial scales than an optical period. This statistical behavior can be mathematically captured quite generally via a *spatiotemporal correlation function*

$$\Lambda(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2)=\left\langle U(\mathbf{r}_1,t_1)\cdot U^*(\mathbf{r}_2,t_2)\right\rangle_{\mathbf{r},t} \tag{4.12}$$

The average < > is performed both temporally and spatially, as indicated by the subscripts **r** and t, via

$$\begin{aligned}\left\langle U(\mathbf{r}_1,t_1)\cdot U^*(\mathbf{r}_2,t_2)\right\rangle_t &= \lim_{T\to\infty}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} U(\mathbf{r}_1,t_1)\cdot U^*(\mathbf{r}_2,t_2)\,dt_1dt_2\\ \left\langle U(\mathbf{r}_1,t_1)\cdot U^*(\mathbf{r}_2,t_2)\right\rangle_{\mathbf{r}} &= \lim_{A\to\infty}\int_A\int_A U(\mathbf{r}_1,t_1)\cdot U^*(\mathbf{r}_2,t_2)\,d^2\mathbf{r}_1d^2\mathbf{r}_2\end{aligned} \tag{4.13}$$

$\frac{1}{T^2}$ $\frac{1}{A^2}$

Often, in practice we deal with fields that are both *stationary* (in time) and *statistically homogeneous* (in space). Stationary means that the statistical properties of the field (e.g. the average, higher order moments) do not depend on the origin of time. Similarly, for statistically homogeneous fields, their properties do not depend on the origin of space.

Under these circumstances, the dimensionality of the spatiotemporal correlation function Λ decreases by half if the fields are stationary and statistically homogeneous, i.e.

$$\begin{aligned}\Lambda(t_1,t_2)&=\Lambda(t_2-t_1)\\ \Lambda(\mathbf{r}_1,\mathbf{r}_2)&=\Lambda(\mathbf{r}_2-\mathbf{r}_1)\end{aligned} \tag{4.14}$$

Wide-sense

Thus, the spatiotemporal correlation function becomes

$$\Lambda(\boldsymbol{\rho},\tau)=\left\langle U(\mathbf{r},t)\cdot U^*(\mathbf{r}+\boldsymbol{\rho},t+\tau)\right\rangle_{\mathbf{r},t} \tag{4.15}$$

Note that $\Lambda(\mathbf{0},0) = \langle U(\mathbf{r},t) \cdot U^*(\mathbf{r},t) \rangle_{\mathbf{r},t}$ represents the average irradiance of the field, which is a real quantity. However, in general, Λ is complex. Let us define a normalized version of Λ, referred to as the *spatiotemporal degree of correlation*

$$\alpha(\boldsymbol{\rho},t) = \frac{\Lambda(\boldsymbol{\rho},\tau)}{\Lambda(\mathbf{0},0)}. \tag{4.16}$$

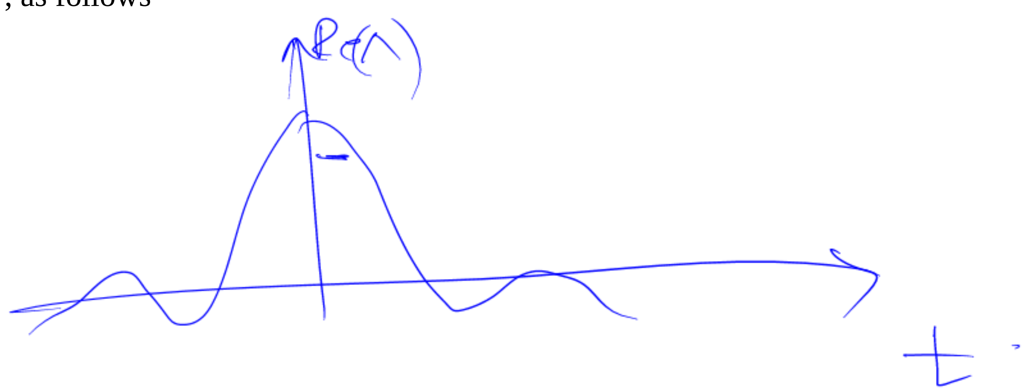
It can be shown that for stationary fields $|\Lambda|$ attains its maximum at $\mathbf{r} = 0, \quad t = 0$, thus

$$0 < |\alpha(\rho,\tau)| < 1 \tag{4.17}$$

We can define an area $A_C \propto \rho_C^{\,2}$ and length $l_c = c\tau_C$, over which $|\alpha(\rho_C,\tau_C)|$ maintains a significant value, say $|\alpha| > 1/2$, which defines a *coherence volume*

$$\boxed{V_C = A_C \cdot l_C.} \tag{4.18}$$

This *coherence volume* determines the average domain size over which the fields can be considered correlated. In general an extended source, such as an incandescent filament, may have spectral properties that vary from point to point. Thus, it is convenient to discuss spatial correlations at each frequency ω., as follows



4.3 Spatial correlations of monochromatic light

By taking the Fourier transform of Eq. 15 with respect to time, we obtain what is referred to as the *cross-spectral density*

$$\begin{aligned} W(\boldsymbol{\rho},\omega) &= \int \Lambda(\boldsymbol{\rho},\tau)\cdot e^{i\omega\tau}\, d\tau \\ &= \left\langle U(\mathbf{r},\omega)\cdot U^*(\mathbf{r}+\boldsymbol{\rho},\omega)\right\rangle_{\mathbf{r}} \end{aligned} \tag{4.19}$$

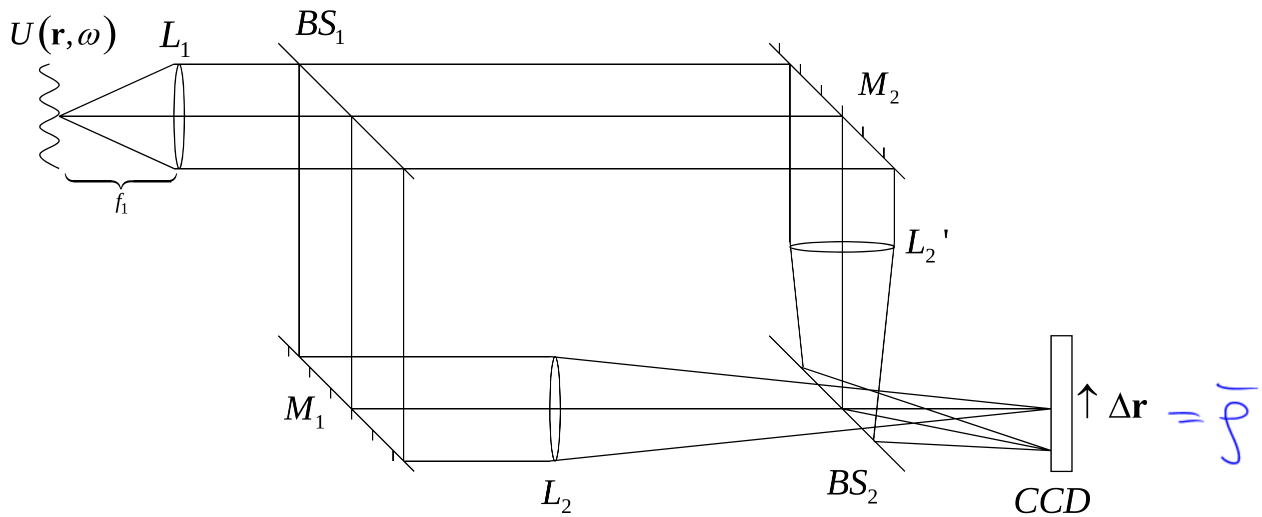


Figure 4-3. Interferometry with spatially extended fields.

An experimental situation when the correlation function defined in Eq. 19 becomes relevant is illustrated in Fig. 3 via an imaging Mach-Zehnder interferometer. Figure 3 depicts an interferometric system where the monochromatic field $U(\mathbf{r},\omega)$ is split in two replicas that are further re-imaged at the CCD plane via two 4f lens systems, which induce a spatial shift $\Delta\mathbf{r}$. The question of practical interest here is: to what extent do we observe fringes, or, more quantitatively, **what is the fringe contrast if we vary $\Delta\mathbf{r}$? For each value of $\Delta\mathbf{r}$** , the CCD records an *average* intensity that has the form

$$\begin{aligned} I(\Delta\mathbf{r}) &= \int \left| U(\mathbf{r},\omega) + U(\mathbf{r}+\Delta\mathbf{r},\omega) \right|^2 d^2\mathbf{r} \\ &= \int 2I(\mathbf{r},\omega)\, d^2\mathbf{r} + \int 2\,\mathrm{Re}\left[U(\mathbf{r},\omega)\cdot U^*(\mathbf{r}+\Delta\mathbf{r},\omega) \right] d^2\mathbf{r} \end{aligned} \tag{4.20}$$

where we assumed that the interferometer splits the light equally on the two arms. It can be seen that the real part of $W(\rho,\omega)$ as defined in Eq. 19 can be measured experimentally using the arrangement in Fig. 3. Multiple CCD exposures are necessary corresponding to each $\Delta\mathbf{r}$. Once the intensity of each beam $I(\mathbf{r},\omega)$ is measured separately, each CCD image is averaged spatially, which provides the value W at a particular shift $\Delta\mathbf{r}$. The complex *degree of spatial correlation* at frequency ω is defined as

$$\beta(\boldsymbol{\rho},\omega) = \frac{W(\boldsymbol{\rho},\omega)}{\left| W(\mathbf{0},\omega) \right|}. \tag{4.21}$$

Note that $W(\mathbf{0},\omega)$ is nothing more than the average optical spectrum of the field, at

$$\begin{aligned} W(\mathbf{0},\omega) &= \left\langle U(\mathbf{r},\omega)\cdot U^*(\mathbf{r}+\Delta\mathbf{r},\omega) \right\rangle_{\mathbf{r}} \\ &= \left\langle S(\omega) \right\rangle_{\mathbf{r}} \end{aligned} \tag{4.22}$$

$\Delta r = 0$.

Again, it can be shown that $|\beta| \in [0,1]$, where the extremum values of $|\beta| = 0$ and $|\beta| = 1$ correspond to complete lack of spatial coherence and full coherence respectively. The area over which $|\beta|$ maintains a significant value defines the coherence area at frequency ω,

$$A_C = D, \quad \text{for which } \left| \beta(\boldsymbol{\rho},\omega) \right| \Big|_{\rho_x \rho_y \subset D} < \frac{1}{2} \tag{4.23}$$

Convention

$\Delta x \sim \frac{1}{\langle \Delta k_x^2 \rangle}$

Since $W(\boldsymbol{\rho},\omega)$ is a spatial correlation function, it can be expressed through a *spatial power spectrum*, via a Fourier transform

$$\begin{aligned} P(\mathbf{k},\omega) &= \iint W(\boldsymbol{\rho},\omega)\cdot e^{i\mathbf{k}\boldsymbol{\rho}} d^2\boldsymbol{\rho} \\ W(\boldsymbol{\rho},\omega) &= \iint \tilde{W}(\mathbf{k},\omega)\cdot e^{-i\mathbf{k}\boldsymbol{\rho}} d^2\mathbf{k} \end{aligned} \tag{4.24}$$

The spatial correlation function $W(\boldsymbol{\rho},\omega)$ can also be determined from measurements of power spectrum, as shown in Fig. 4.

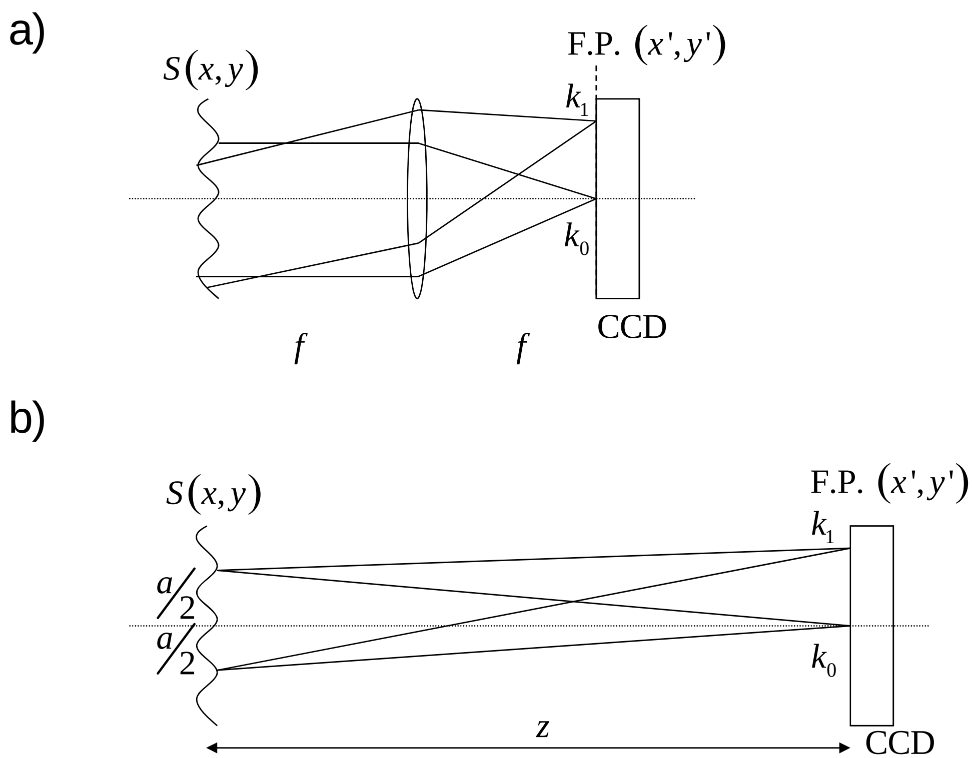


Figure 4-4. Fourier transforming the source field via a lens (a) and far field propagation in free space (b).

Recall that both the far field propagation in free space and propagation through a lens can generate the Fourier transform of the source field, as illustrated in Fig. 4,

$$\tilde{U}(\mathbf{k},\omega)=\int U(\mathbf{r},\omega)\cdot e^{-i\mathbf{kr}}d^2\mathbf{r}, \qquad (4.25)$$

The CCD detects the spatial power spectrum, $P(\mathbf{k},\omega)=\left|\tilde{U}(\mathbf{k},\omega)\right|^2$.

In Eq. 25, the frequency component $\mathbf{k}$ depends on the focal distance for the situation in Fig. 4 and the propagation distance z (Fig. 4b); respectively

$$\begin{aligned}\mathbf{k}&=\frac{2\pi}{\lambda f}(x',y')\\ \mathbf{k}&=\frac{2\pi}{\lambda z}(x',y')\end{aligned} \qquad (4.26)$$

Another question one may ask is: what is the spatial coherence of the field upon propagation? For extended sources that are far away from the detection plane, as in Fig. 4b, the size of the source may have a significant effect on the Fourier transform in Eq. 25. This effect becomes obvious if we replace the source field U with its spatially truncated version, $\underline{U}$, to indicate the finite size of the source

$$\underline{U}(\mathbf{r},\omega)=U(\mathbf{r},\omega)\cdot\Pi\left(\frac{\mathbf{r}}{a}\right), \qquad (4.27)$$

where function Π is the typical *2D rectangular function*, there denoting a square of side a. Thus, the far field becomes

$$\underline{\tilde{U}}(\mathbf{k},\omega)=a^2\tilde{U}(\mathbf{k},\omega)*\text{sinc}(a\mathbf{k}), \qquad (4.28)$$

where *sinc* is our common 2D function, $\mathrm{sinc}(\mathbf{k})=\frac{\sin k_x}{k_x}\cdot\frac{\sin k_y}{k_y}$. Thus, the field across detection plane *(x', y')* is smooth over scales given by the width of the *sinc* function. Along x', this distance, x_c', is given by

$$\begin{aligned}\frac{2\pi}{a}&=k_x\\ &=\frac{2\pi}{\lambda z}\cdot x_c'\end{aligned}\tag{4.29}$$

We can conclude that the coherence area of the field generated by the source in the far zone is

$$\begin{aligned}A_C&=x_c^{\ 2}=\\ &=\frac{\lambda^2}{\Delta\Omega},\end{aligned}\tag{4.30}$$

where Ω is the solid angle subtended by the source. This simple relationship allowed Michelson to measure interferometrically the angle subtended by stars. For example, the Sun subtends and angle $\theta\simeq 10\,\mathrm{mrad}$, i.e. $\Omega\simeq 10^{-4}\mathrm{srad}$. Thus, for the green radiation, $\lambda=550\mathrm{nm}$, the coherence area at the surface of the Earth is $A_C^{\mathrm{Sun}}=50\cdot 50\mu\mathrm{m}^2$. For angularly smaller sources, far field spatial coherence is correspondingly higher. **This is the essence of the Van Cittert-Zernike theorem, which states that the field generated by spatially incoherent sources gains coherence upon propagation.** This is the result of free-space propagation acting as a (spatial) low-pass filter. Thus, from the properties of Fourier transforms, we infer that higher spatial coherence at frequency ω, i.e. a broader $W(\rho,\omega)$, can be obtained by narrower $\tilde{W}(\mathbf{k},\omega)$. When dealing with extended sources, it is common practice in the laboratory to perform low pass filtering on $\tilde{W}(\mathbf{k},\omega)$. This procedure is called, not surprisingly, *spatial filtering*, and is illustrated in Fig. 5.

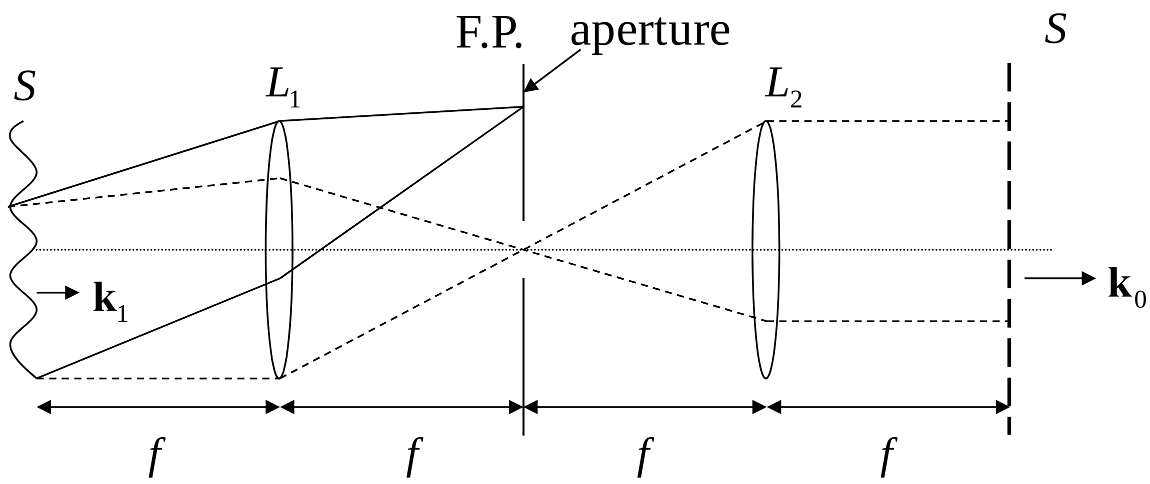


Figure 4-5. Spatial filtering via a 4f system.

In Fig. 5, the extended, broad band source S emits light at a multitude of frequencies ω, and spatial frequencies **k**. At a given frequency ω, lens L_1 performs the spatial Fourier transform at the plane F.P. If an aperture is placed at plane FP to block the high frequencies, the field reconstructed by lens L_2 at plane S' (conjugate to S') approximates a plane wave of wave vector $\mathbf{k}_0$. With this procedure, from an extended source, we obtain a high spatial coherence field. Of course, this procedure is lossy, as the energy carried by the high spatial frequency is lost. Asymptotically, closing down the aperture to a small pinhole, (δ-function in **k** domain) generates a field that approaches a plane wave at plane S' ($\delta \to 1$). Conversely, it should be pointed out that all sources exhibit spatial coherence at least at the scale of the wavelength. This is easily understood by noting that a δ-correlated source, $W(\rho) = \delta(\rho)$ requires $\tilde{W}(\mathbf{k})$

infinitely broad, i.e. $\tilde{W}(\mathbf{k})=1$. Clearly, this is impossible, because a planar source can only emit in a 2π srad solid angle. Thus the minimum coherence area for an arbitrary source is of the order of (Eq. 30)

$$A_C^{\min} \simeq \frac{\lambda^2}{2\pi}. \qquad (4.31)$$

4.4 Temporal correlations of plane waves

Let us now have the discussion symmetric to that in section 3, where we investigate the temporal correlations of fields at a particular spatial frequency **k**. Taking the Fourier transform of Eq. 15 with respect to **r**, we obtained the *temporal correlation function*

$$\begin{aligned}\Gamma(\mathbf{k},\tau) &= \iint \Lambda(\rho,\tau)\cdot e^{-i\mathbf{kr}} d^2\mathbf{r} \\ &= \langle U(\mathbf{k},t)\cdot U^*(\mathbf{k},t+\tau)\rangle_t\end{aligned} \quad (4.32)$$

The autocorrelation function Γ is relevant in interferometric experiments of the type illustrated in Fig. 6.

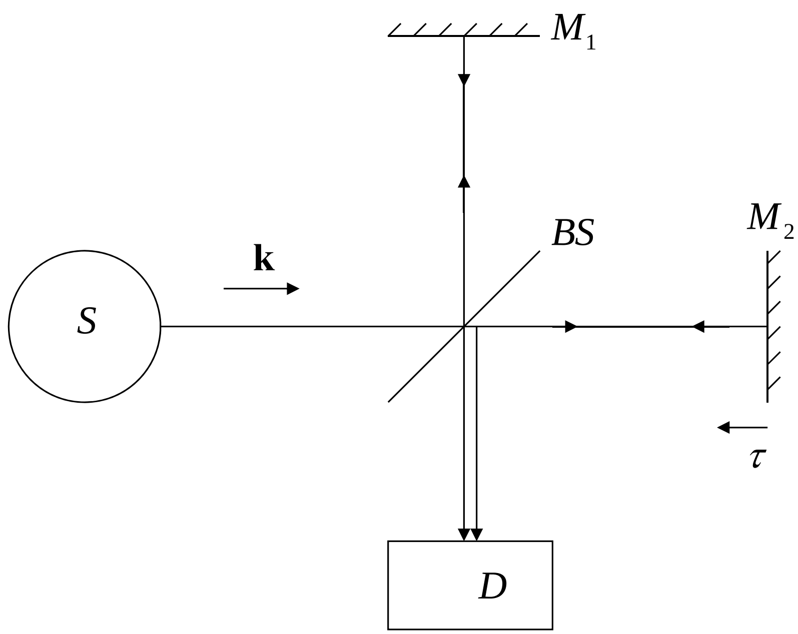


Figure 4-6. Michelson Interferometer.

Figure 6 shows a Michelson interferometer where a plane wave from the source is split in two by the beam splitter and subsequently recombined via reflections on mirrors M_1, M_2. The intensity at the detector has the form (we assume 50/50 beam splitter)

$$\begin{aligned} I(\tau) &= \left\langle \left| U(\mathbf{k},t) + U^*(\mathbf{k},t+\tau) \right|^2 \right\rangle_t \\ &= 2I(\mathbf{k}) + 2\,\mathrm{Re}\left\langle U(\mathbf{k},t)\cdot U^*(\mathbf{k},t+\tau) \right\rangle \end{aligned} \tag{4.33}$$

Thus the real part of $\Gamma(\mathbf{k},\tau)$ is obtained by varying the time delay between the two fields. This delay can be controlled by translating one of the mirrors.

The *complex degree of temporal correlation* at spatial frequency **k** is defined as

$$\gamma(\mathbf{k},\tau) = \frac{\Gamma(\mathbf{k},\tau)}{\left|\Gamma(\mathbf{k},0)\right|} \tag{4.34}$$

Note that $\Gamma(\mathbf{k},0)$ represents the irradiance of the field, i.e.

$$\begin{aligned} \Gamma(\mathbf{k},0) &= \left\langle U(\mathbf{k},t)\cdot U^*(\mathbf{k},t) \right\rangle_t \\ &= I(\mathbf{k}) \end{aligned} \tag{4.35}$$

The complex degree of temporal correlation has the similar property with its spatial counterpart β, i.e.

$$0 < \left|\gamma(\mathbf{k},\tau)\right| < 1 \tag{4.36}$$

Similarly, the *coherence time* is defined as the maximum time delay between the fields for which $|\gamma|$ maintains a significant value, say ½.

The temporal correlation Γ is the Fourier transform of the power spectrum,

$$\Gamma(\mathbf{k},\tau)=\int_{-\infty}^{\infty} S(\mathbf{k},\omega)\cdot e^{-i\omega\tau}d\omega$$
$$S(\mathbf{k},\omega)=\int_{-\infty}^{\infty} \Gamma(\mathbf{k},\tau)\cdot e^{i\omega\tau}d\omega \tag{4.37}$$

Thus Γ can be determined via spectroscopic measurements, as shown in Fig. 7

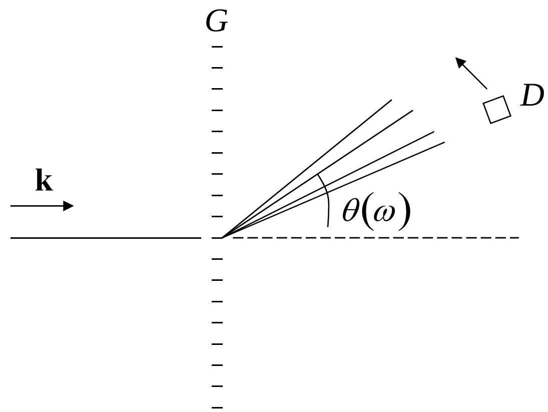


Figure 4-7. Spectroscopic measurement using a grating: G grating, D detector, $\theta(\omega)$ diffraction angle. The dashed line indicates the undiffracted order (zeroth order).

By using a grating (or a prism), we can disperse different colors at different angles, such that a rotating detector can measure $S(\omega)$ directly. Let us assume a field of Gaussian spectrum centered at frequency ω_0, and having the r.m.s. width $\Delta\omega$,

$$S(\omega)=S_0\cdot e^{-\left(\frac{\omega-\omega_0}{\sqrt{2}\Delta\omega}\right)^2}, \tag{4.38}$$

where S_0 is a constant. The autocorrelation function is also a Gaussian, modulated by a sinusoidal function, as a result of the Fourier shift theorem

$$\Gamma(\tau) = \Gamma_0 \cdot e^{-\left(\frac{\Delta\omega\tau}{\sqrt{2}}\right)^2} \cdot e^{i\omega_0\tau} \tag{4.39}$$

From Eq. 39, we see that if we define the r.m.s. width of Γ as the coherence time, we obtain

$$\tau_C \simeq \frac{1}{\Delta\omega}, \tag{4.40}$$

and the coherence length

$$\begin{aligned} l_C &= c\tau_C \\ &\propto \frac{\lambda^2}{\Delta\lambda} \end{aligned} \tag{4.41}$$

The coherence length depends on the spectral bandwidth in an analog fashion to the coherence area dependence on solid angle (Eq. 27). This is not surprising as both types of correlations depend on their respective frequency bandwidth (the spatial frequency spectrum is, up to a constant, the angular spectrum, which in 2D becomes solid angle). The coherence length values can vary broadly, from kilometers for a narrow bad laser, to microns for LEDs and white light. Figure 7 shows qualitatively the relationship between $\Gamma(\tau)$ and $S(\omega)$.

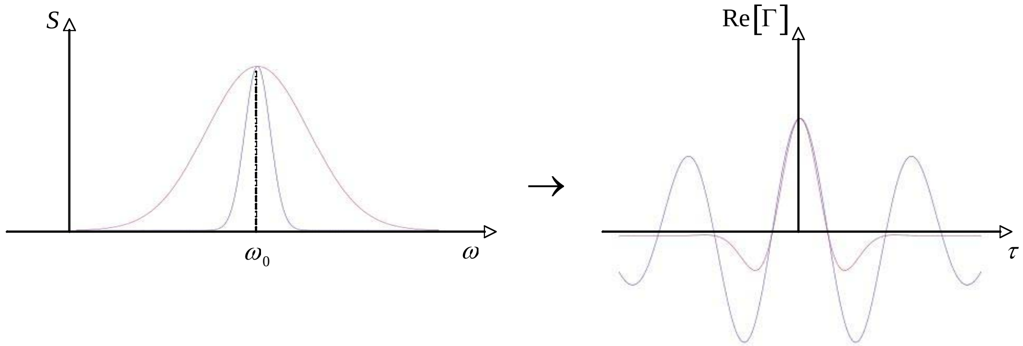


Figure 4-8.

Of course, using narrow band filters has the effect of enlarging the coherence length of the field.