Fourier Optics

1. Superposition principle

- The output of a sum of inputs equals the sum of the respective outputs
- *Input* examples*:*
  - force applied to a mass on a spring
  - voltage applied to a RLC circuit
  - optical field impinging on a piece of tissue
  - etc.
- *Output* examples:
  - Displacement of the mass on the spring
  - Transport of charge through a wire
  - Optical field scattered by the tissue
- Consequence: ***a complicated problem (input) can be broken into a number of simpler problems***

1. Superposition principle

- Two fields incident on the medium
- Two choices:
  - i) add the two inputs, $U_1 + U_2$, and solve for the output
  - ii) find the individual outputs, U'_1, U'_2 and add them up, $U'_1 + U'_2$
- Choice ii) employs the superposition principle
- We can decompose signals further:
  - Green's method
    - Decompose signal in delta-functions
  - Fourier method
    - Decompose signal in sinusoids

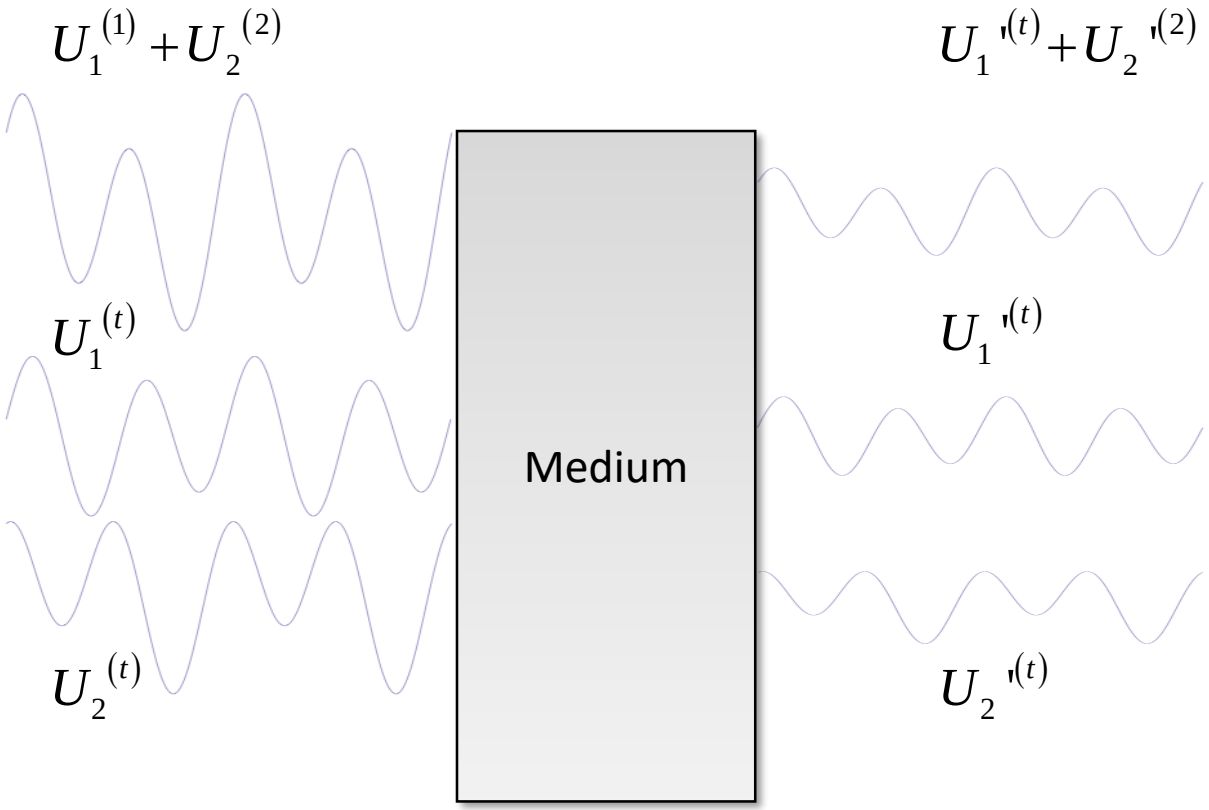


Figure 1.1. *The superposition principle. The response of the system (e.g. a piece of glass) to the sum of two fields, U1+U2, is the same as the sum of the output of each field, U1'+U2'.*

1.1. Green's function method

- Decompose input signal into a series of infinitely thin pulses
  - *Dirac delta function:*

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

  - *Normalization:*

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

  - *Sampling:*

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

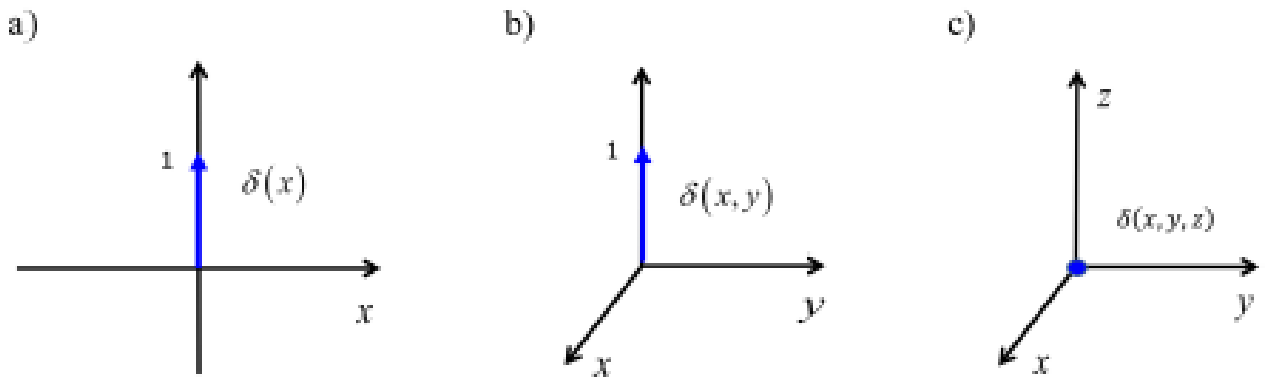
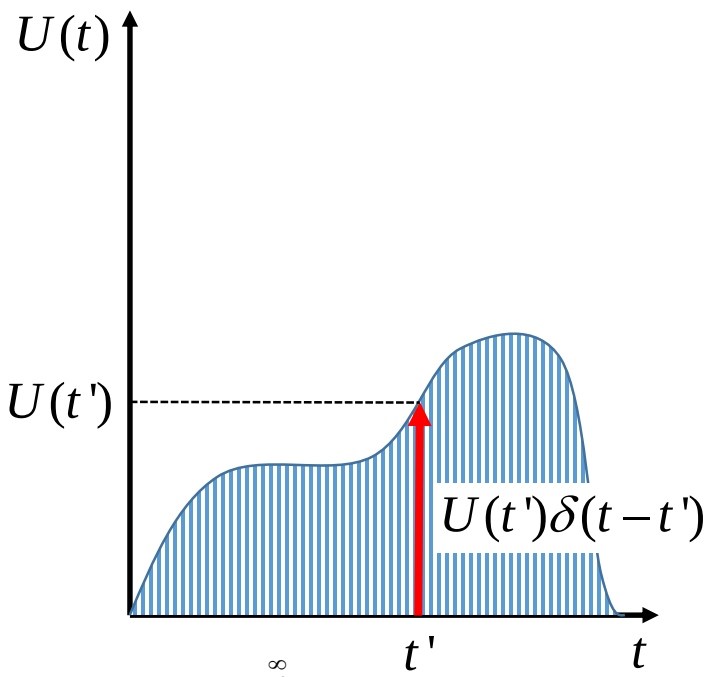


Figure 1.2. *Delta function in 1D(a), 2D(b), 3D(c)*

- *Also note:* $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$

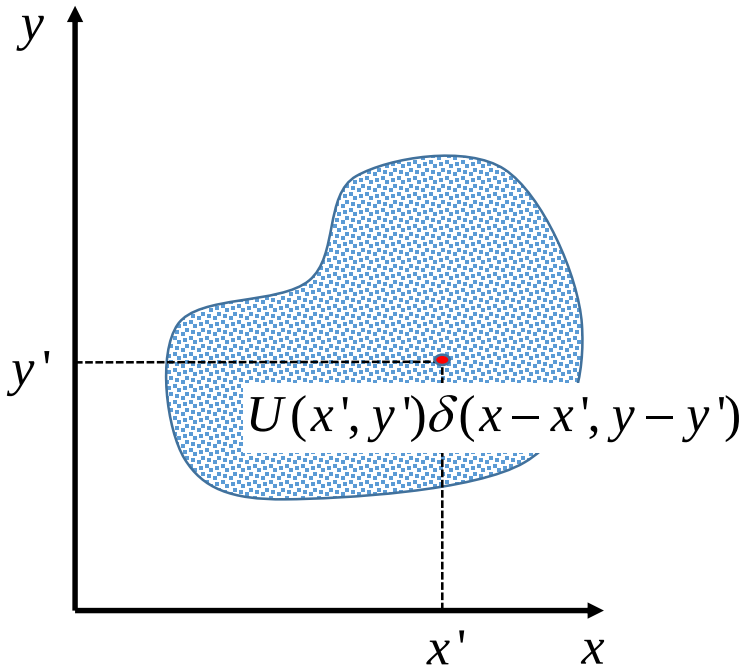
1.1. Green's function method

a)1D case



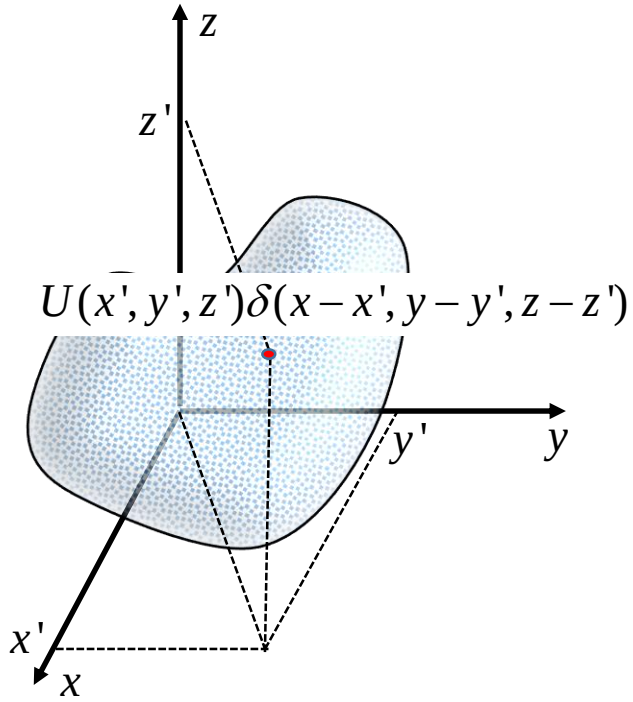
$$U(t)=\int_{-\infty}^{\infty}U(t')\delta(t-t')dt'$$
$$=U(t)\circledv\delta(t)$$

b)2D case



$$U(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(x',y')\delta(x-x',y-y')dx'dy'.$$

c)3D case



$$U(x,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(x',y',z')\delta(x-x',y-y',z-z')dx'dy'dz'.$$

***Figure 1.3**. 1D (a), 2D (b), and 3D (c) signals can be described as an ensemble of impulses. The delta-functions have their amplitudes equal to the signal evaluated at the position of the delta function, namely, $U(t')$, $U(x',y')$, $U(x',y',z')$.*

If shifted by *a* $U(t)\circledv\delta(t-a)=U(t-a)$ Note we denote *convolution* as $\circledv$

1.2. Fourier transform method

- Any plot can be represented by a series of sinusoids
  - Solving linear problem for one sinusoid as input is easy!
  - Output is summation of all sinusoid responses

Temporal Signals:

$$+ \ldots$$

ω = temporal angular frequency

Spatial Signals:

$$A(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$$

k = (k_x,k_y,k_z) = Angular spatial frequenc

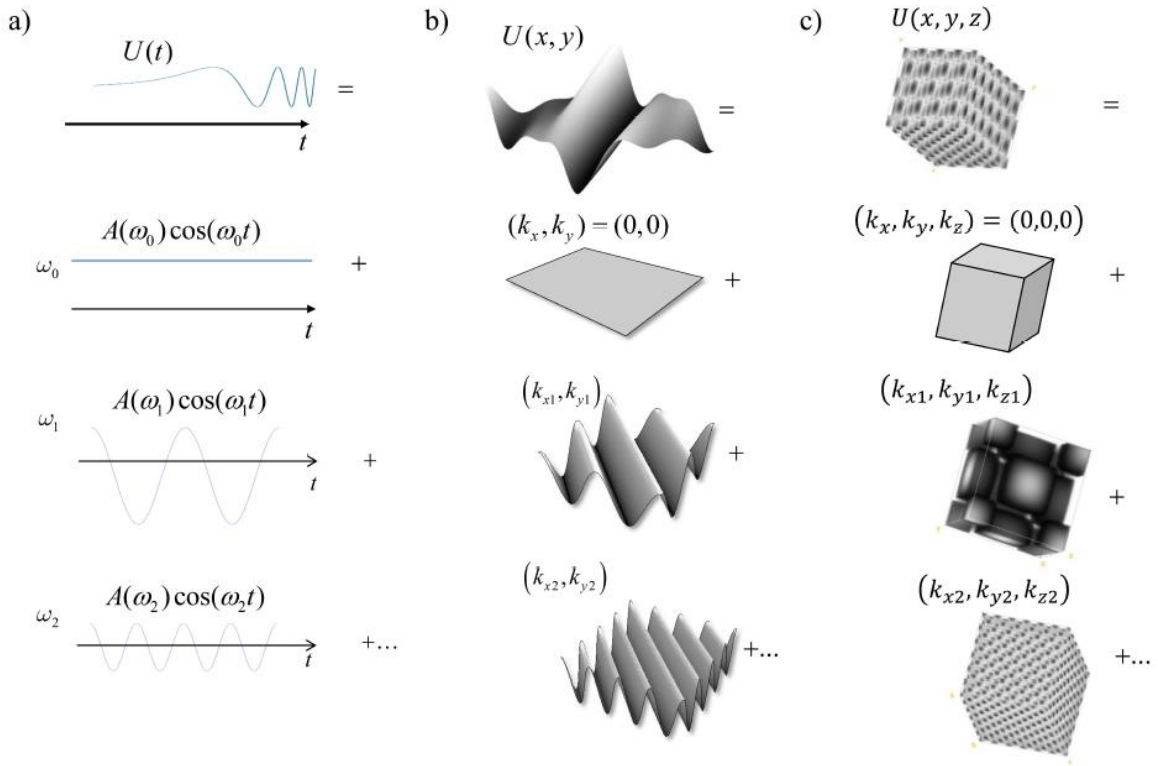


Figure 1.4. *A signal can be decomposed into a sum of sinusoids: 1D (a), 2D (b), 3D (c).*

1.3. Example Problems

- Express as *convolutions* with δ-functions.

$\Pi(x-5)$

$\Gamma(\mathrm{x}-2)+\mathrm{sinc}(\frac{2x}{3}+5)$

$\Pi(\frac{3x}{2}+3)-\Pi(\frac{5x}{2}-2)$

$\sum_{n=1}^{5}\Lambda(\frac{x}{2}-2n)$

- Prove the *sampling property* of the delta function.

$$\mathrm{f}(x)\delta(x-a)=f(a)\delta(x-a).$$

- Solve the integrals.

$\int_{-\infty}^{\infty}\delta(x)\left[x^2+x+1\right]dx$

$\int_{-\infty}^{\infty}\delta(x-5)\left[x^2+x+1\right]dx$

$\int_{-\infty}^{\infty}\delta(x)\left[e^{ixb}\right]dx$ *b real constant*

$\int_{-\infty}^{\infty}\delta(x-5)\left[e^{ixb}\right]dx$ *b real constant*

2. Linear Systems

- $f(t)$ is an arbitrary input, and $g(t)$ is the output of our system.

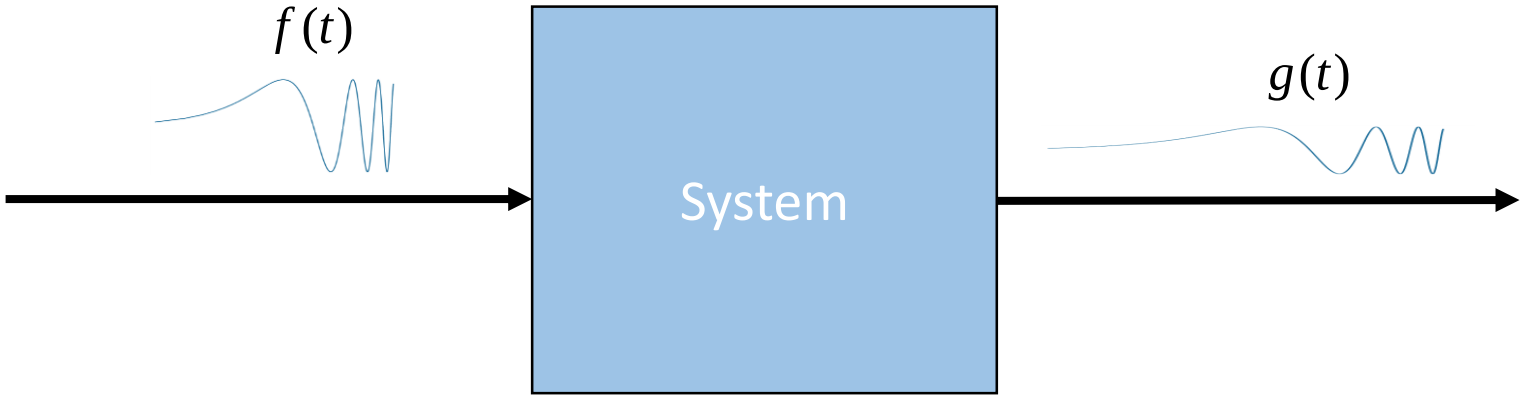


Figure 2.1. *Input and output of a system.*

- The system is characterized by a mathematical *operator*, *L*, such that $L\left[f(t)\right] = g(t)$.

- Since the system is **linear**, this operator is a **complete characterization** of the system!

2.1. Linearity

- What makes a system Linear?
  - A system is linear if the system's response to a linear combination of *inputs* is a linear combination of *outputs*.

$$L\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 L\left[f_1(t)\right] + a_2 L\left[f_2(t)\right]$$
$$= a_1 g_1(t) + a_2 g_2(t),$$

Where a_1 and a_2 are arbitrary scalars

  - Remember from earlier $L[f_1(t)] = g_1(t)$ & $L[f_2(t)] = g_2(t)$
  - L is a linear operator

2.1. Linearity

- What is the main takeaway of linear systems?
  - *Impulse Response*
    - *First express function as sum of impulses*

$$f(t) = \int_{-\infty}^{\infty} f(t')\delta(t-t')dt'$$
$$= \sum_i f(t_i)\delta(t-t_i)(t_{i+1}-t_i).$$

    - *Systems response to input f(t) is the sum or integral of outputs*

$$L\left[f(t)\right] = \int_{-\infty}^{\infty} f(t')L\left[\delta(t-t')\right]dt'$$

    - Since we deal with finite signals the system is fully characterized by its *impulse response, h(t,t')*

$$h(t,t') = L\left[\delta(t-t')\right].$$

2.2. Shift invariance

- For *linear shift invariant (LSI) systems*, the response to a shifted impulse is the shifted impulse response

$$L\left[\delta\left(t-t'\right)\right]=h\left(t,t'\right)$$
$$=h\left(t-t'\right).$$

  - This means the *shape* of the impulse response is time independent!
  - This allows us to calculate the output g(t) with an input f(t)!

$$g\left(t\right)=\int_{-\infty}^{\infty} f\left(t'\right)h\left(t-t'\right)dt'.$$

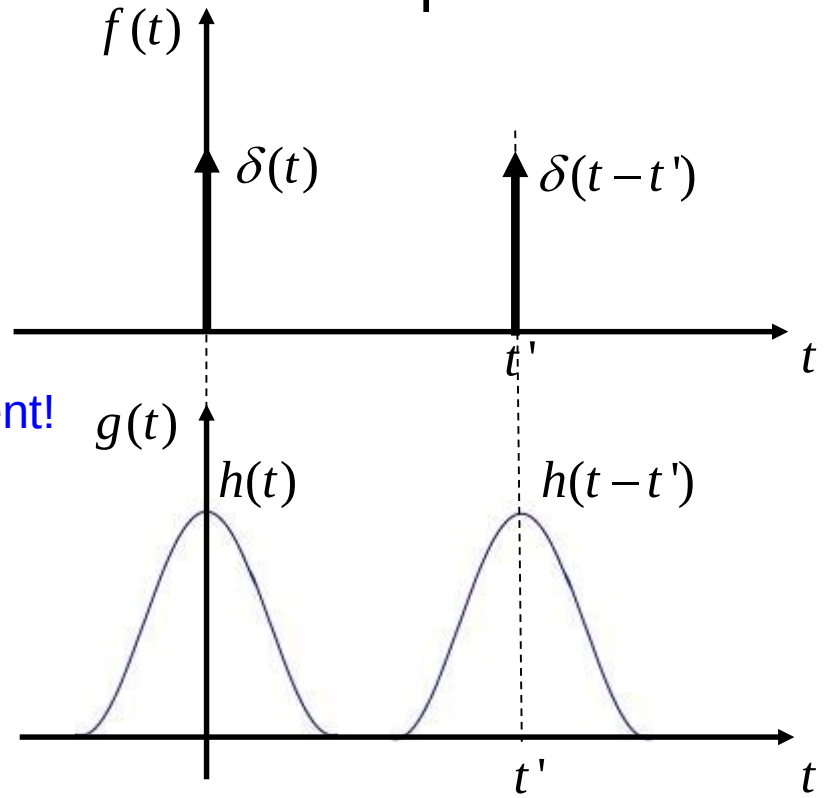


Figure 2.2. *In a linear shift-invariant system, the response to a pulse shifted by t' is the impulse response shifted by the same amount, t'.*

2.3. Causality

- Can you become my favorite student without coming to class?
  - NO! The effect cannot precede its cause ☺
- An output **cannot** precede its input
  - Mathematically if $f(t) = 0$, for $t < t_0$

    then $g(t) = 0$, for $t < t_0$.
- Output can be written as: $g(t) = \int_{t_0}^{\infty} f(t - t')h(t')dt'$, and *h(t) = 0 when t < t_0*

  Notice t_0 is our lower bound!

- An LSI system is causal *if and only if* the impulse response, *h(t)*=0 when t < 0

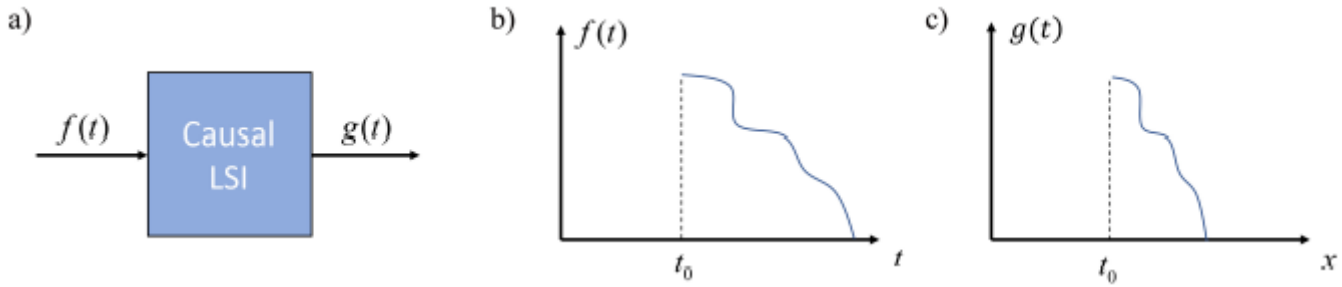


Figure 2.3. *Input (b) and output (c) for a causal system (a).*

2.4. Stability

- What defines a stable system?
  - A system is stable if it responds to a bounded input *f(t)* with a bounded output *g(t)*
  - Mathematically: $\text{if } |f(t)| < b$ *b* is a constant and α is a *system specific* constant

    $\text{then } |g(t)| < \alpha b,$
  - How to find α
    - From our definition of impulse response $|g(t)| = \left| \int_{-\infty}^{\infty} f(t-t')h(t')dt' \right|$

      $\leq b \int_{-\infty}^{\infty} |h(t')| dt'.$
    - This proves that if the system is stable, then the impulse response is *absolute-integrable*.

      $$\alpha = \int |h(t)| dt < \infty$$

- Therefore, we can conclude that a linear system is stable *if and only if* its impulse response is modulus-integrable

  $$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty \, .$$

2.5. Example Problems

- The response of an LSI system to $\Gamma(x+\mathrm{a})$ is $\Pi(\mathrm{x})$
  - For what values of *a* is the system causal?
  - If *a* = 5, find the response to the input $f(x)=\mathrm{e}^{\frac{-x^2}{2}}$
  - What is the transfer function of the system?
  - Is the system stable?
  - Is the system causal?
- A systems response to an input *f(x)* is $g(x)=f(x+b)+f^2(x+b)$.
  - Is the system linear?
  - Is the system shift invariant?
  - For what values of *b* is the system causal?
- Which of the following impulse responses correspond to stable and causal systems?
  - $h(x)=\Gamma(x-4)$.
  - $h(x)=\Pi(x+2)$
  - $h(x)=e^{-x^2}$
  - $h(x)=e^{x}$

3. Spatial and temporal frequencies

3.1. Monochromatic Plane Waves

- Two very important exponentials we will use often
  - $e^{-i\omega t}$ for temporal variations of a monochromatic field.
  - ω is angular frequency [radians/sec]

  - $e^{ik_x x}$ for spatial variations of a plane wave along the x-axis
  - k_x is *wavenumber* or spatial frequency [radians/meter]
- *Interesting fact*
  - Angular frequency of a HeNe laser is $\omega = 3\times10^{15} rad/s$
  - Taking the real part of the exponential for this laser (cosine term) we get a period of $T = 2\pi/\omega = 2.1\times10^{-15} s$
  - This is roughly 2 femtoseconds!
- *Normally we will visualize the <u>real part</u> of these important exponentials*

3.1. Monochromatic plane waves $e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$

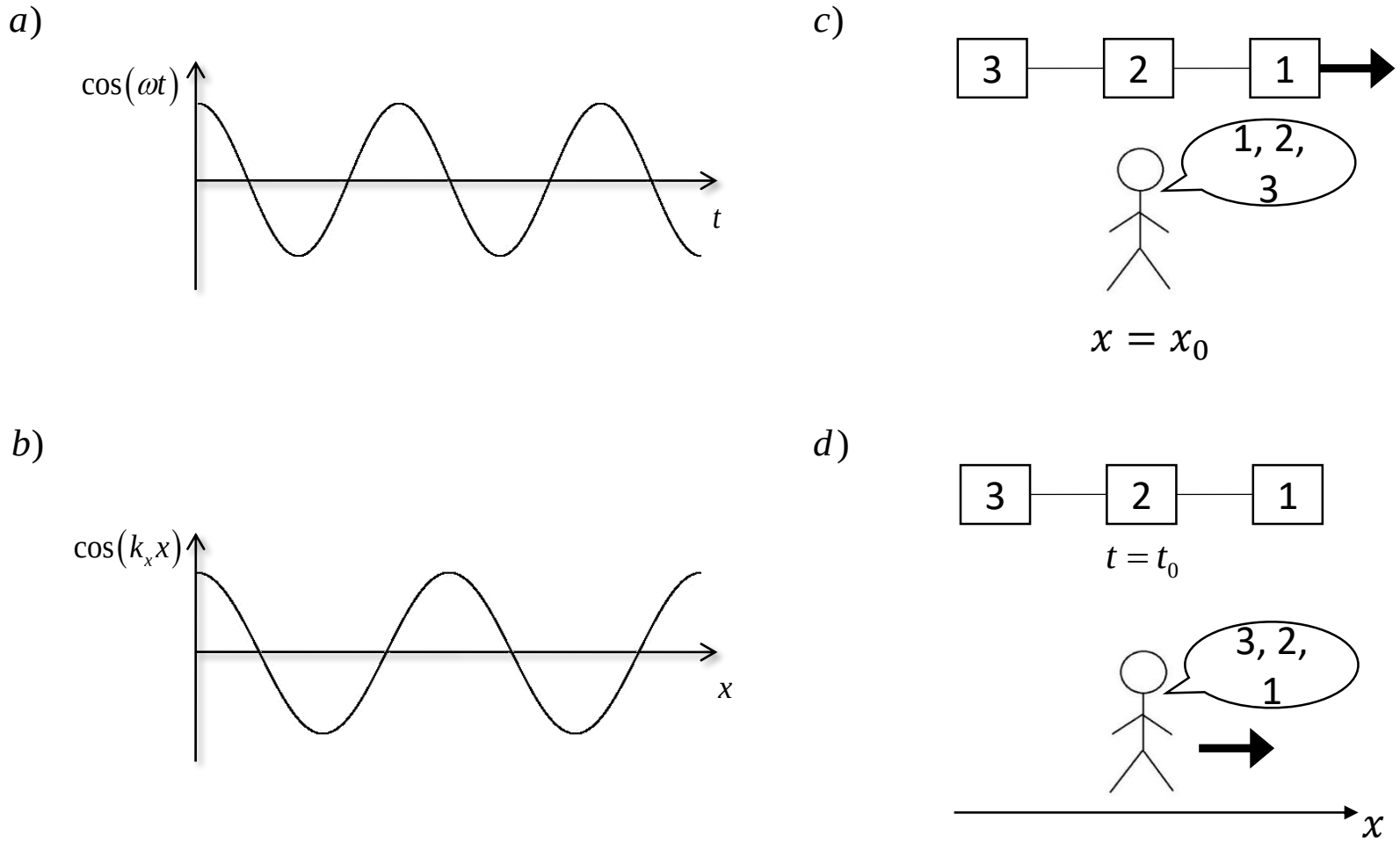


Figure 3.1. *Temporal (a) and spatial (b) variation of (the real part of) a monochromatic plane wave. c) An observer positioned at x_0 counts the cars of a train passing by: 1, 2, 3, …. d) The train is "frozen" in time and now the observer counts the cars while walking in the +x direction: 3, 2, 1.*

3.1. Monochromatic Plane waves

- The observer (c) at a fixed *spatial* position x_0 "sees" the wave passing with a **temporal phase**: $\phi(t) = -\omega t$
  - Observer (c) sees the "train" go by in the order 1,2,3

- The observer (d) "freezes" the wave at t_0 and walks along the x-axis. This gives a **spatial phase**: $\phi(x) = k_x \cdot x$
  - Observer (d) walks by the stopped "train" and sees the order 3,2,1

3.1. Monochromatic Plane Waves

- $e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$ describes a monochromatic plane

- $\mathbf{k}\cdot\mathbf{r}$ is nonzero when the direction of propagation is not parallel to the position vector, **r**

- *Spatial phase* can be written as: $\phi(\mathbf{r}) = \mathbf{k}\cdot\mathbf{r}$ or $\phi(\mathbf{r}) = kr\cos\theta$
  - Notice along direction perpendicular to k there is NO PHASE CHANGE

- Side notes
  - Wavelength is distance between two crests or two troughs
  - Wavelength and wavenumber are related by $\lambda = 2\pi / |\mathbf{k}|$

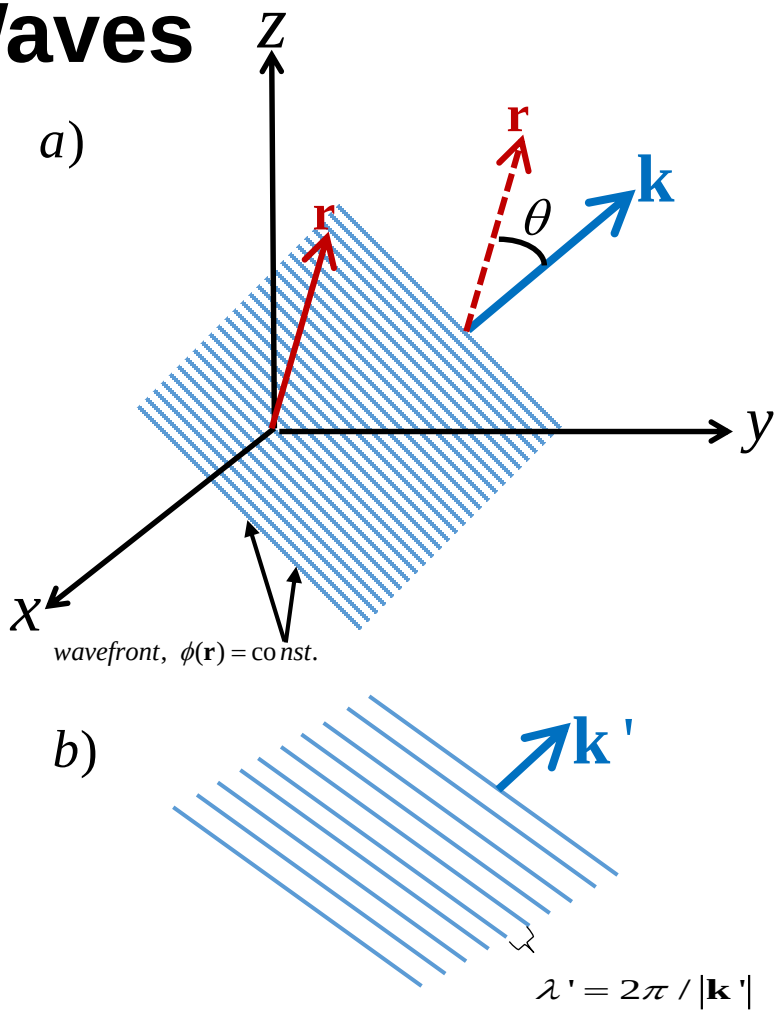


Figure 3.2. *a) Plane wave propagation at an arbitrary angle with respect to the coordinate system. The wavefront is defined by the surface perpendicular to* ***k****, containing points of equal phase. b) Plane wave of longer wavelength than in a).*

3.2. Eigenfunction of a LSI system

- A critical property of LSI systems is the response to a complex exponential is also a complex exponential $L\left(e^{-i\omega t+i\mathbf{kr}}\right)=\alpha e^{-i\omega t+i\mathbf{kr}}$, where α is a constant
  - This is a characteristic of eignefunctions!
  - Proof:
    - $L\left(e^{-i\omega t}\right)=g\left(t\right)$. Where g(t) is the response to the exponential
    - Using shift invariance property $L\left[f\left(t-t'\right)\right]=g\left(t-t'\right)$ we get $L\left[e^{-i\omega\left(t-t'\right)}\right]=g\left(t-t'\right)$.
    - Using linearity property we get $L\left[e^{i\omega t'}\cdot e^{-i\omega t}\right]=e^{i\omega t'}g\left(t\right)$.
    - This holds for any *t* at *t*=0: $g\left(-t'\right)=g\left(0\right)e^{i\omega t'}$.
    - Thus we get our result $L\left(e^{-i\omega t}\right)=\alpha e^{-i\omega t}$,

4. 1D Fourier transform

4.1 Definition and conditions of existence

- For every $\omega \in (-\infty, \infty)$, $f(\omega)$ defines the Fourier transform of $f(t)$:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt.$$

- *Inversion formula*:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t}\, d\omega$$

- Requirements for f to have a Fourier transform:
  - i) f must be modulus-integrable, $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$;
  - ii) f must have a finite number of discontinuities within any finite domain,
  - iii) f must have no infinite discontinuities.

4.1 Definition and conditions of existence

- For any signal that violates $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$, we define a truncated version $\underline{f}$

$$\underline{f}(t) = \begin{cases} f(t), & \text{if } t \in \left(-T/2, T/2\right) \\ 0, & \text{rest} \end{cases}.$$

- Example:

$$\underline{\tilde{f}}(\omega) = \int_{-T/2}^{T/2} \cos(\alpha t) \cdot e^{i\omega t}\, dt$$

$$= \frac{1}{2} \int_{-T/2}^{T/2} \left(e^{i\alpha t} + e^{-i\alpha t}\right) \cdot e^{i\omega t}\, dt = \frac{\sin\left[(\omega+\alpha)T/2\right]}{\omega+\alpha} + \frac{\sin\left[(\omega-\alpha)T/2\right]}{\omega-\alpha}$$

$$= \lim_{T\to\infty} \underline{\tilde{f}}(\omega) = \delta(\omega+\alpha) + \delta(\omega-\alpha).$$

4.2. Significance of the spectral phase

- $f(t)$ can be reconstructed by a superposition of $e^{-i\omega t}$, of different ω.
- Fourier transform:

$$\tilde{f}(\omega) = \left| f(\omega) \right| e^{i\phi(\omega)}$$

  - Amplitude: $\left| f(\omega) \right|$
  - (Spectral) phase: $\phi(\omega)$

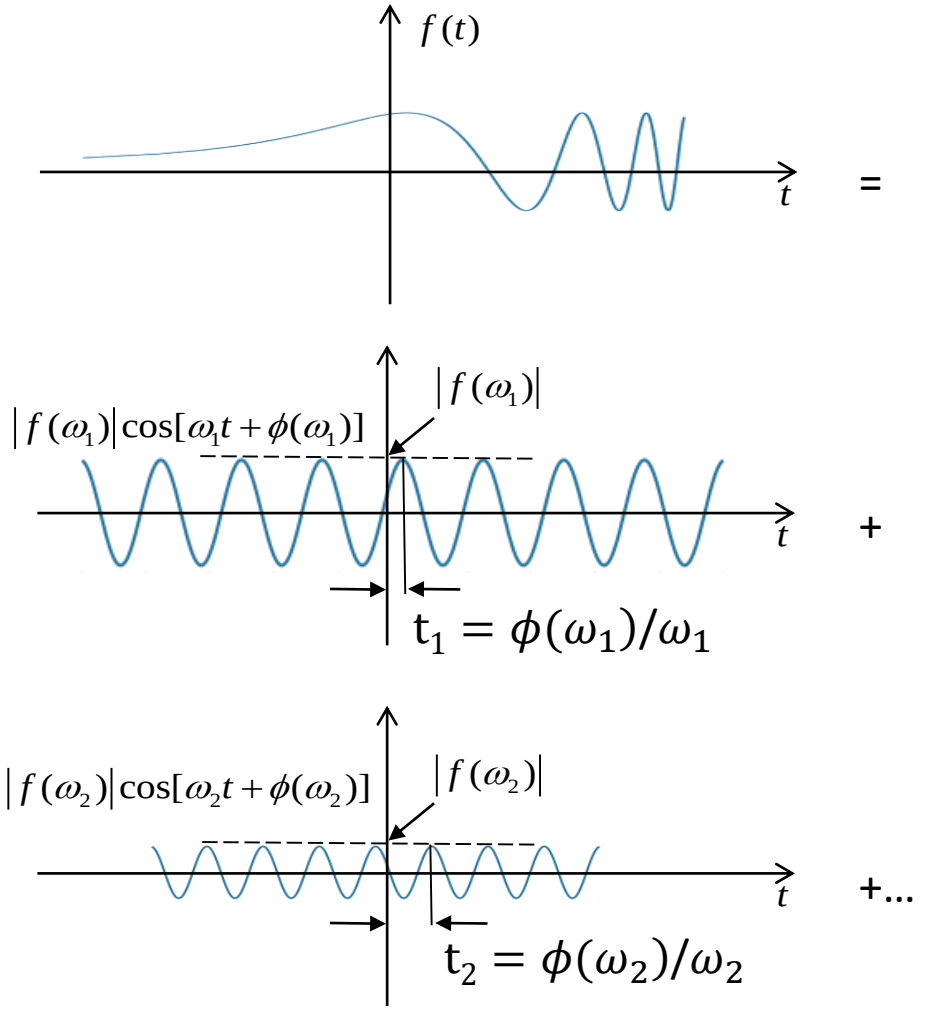


Figure 4.1. *Reconstructing the signal f(t) from is Fourier components, represented here in terms of the real parts. Each Fourier (frequency, harmonic) component is characterized by an amplitude and phase.*

4.2. Significance of the spectral phase

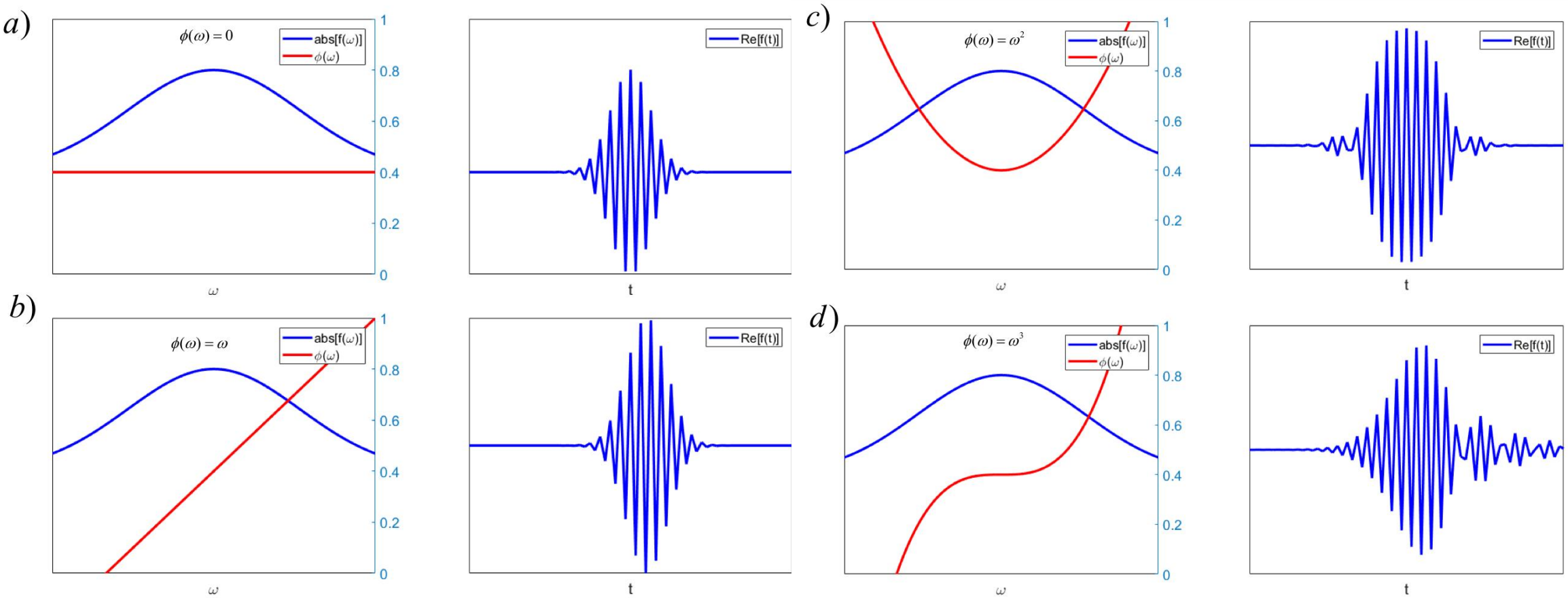


Figure 4.2. *The importance of the spectral phase at constant spectral amplitude. a) Constant spectral phase; b) linear spectral phase yields a signal of the same spread, but shifted in time, c) quadratic spectral phase leads to pulse broadening, d) third order spectral phase yields broadening and asymmetry.*

4.3 Properties of ID Fourier transform

- **Linear:**

$$\Im\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 \tilde{f}_1(\omega) + a_2 \tilde{f}_2(\omega).$$

- a) Central ordinate theorem:

$$\tilde{f}(0) = \int_{-\infty}^{\infty} f(t)\,dt \qquad f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\,d\omega$$

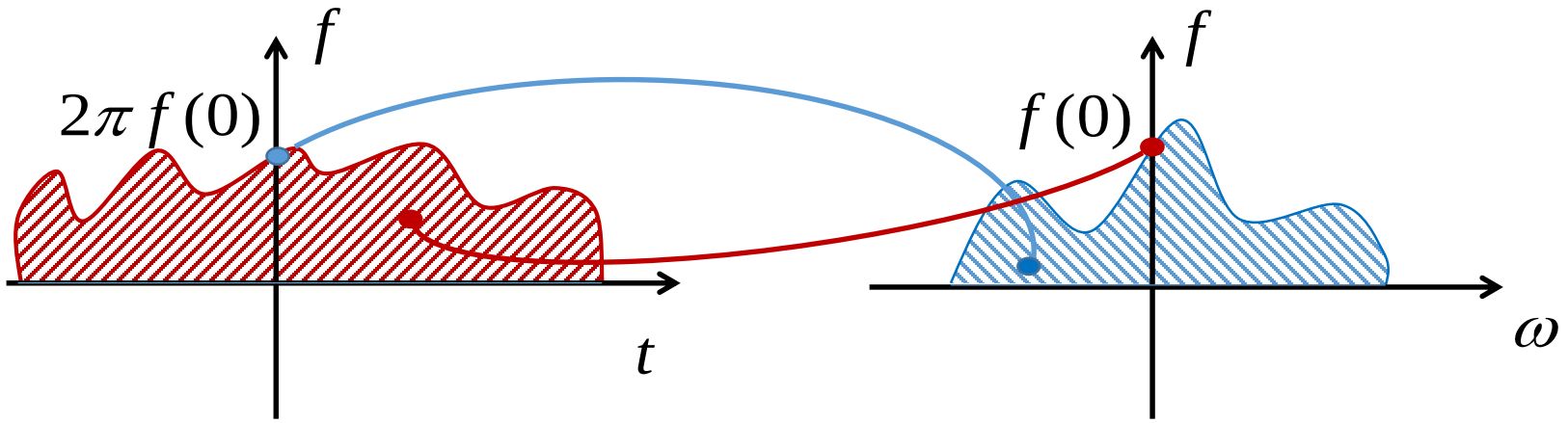


Figure 4.3. *The central ordinate theorem: the integral of a function (its area) in one domain relates to the value at the origin in the other domain.*

4.3 Properties of ID Fourier transform

- b) Shift property:

$$f\left(t-t_0\right) \leftrightarrow \tilde{f}\left(\omega\right)e^{i\omega t_0} \ .$$

$$\tilde{f}\left(\omega-\omega_0\right) \leftrightarrow f\left(t\right)e^{-i\omega t_0}$$

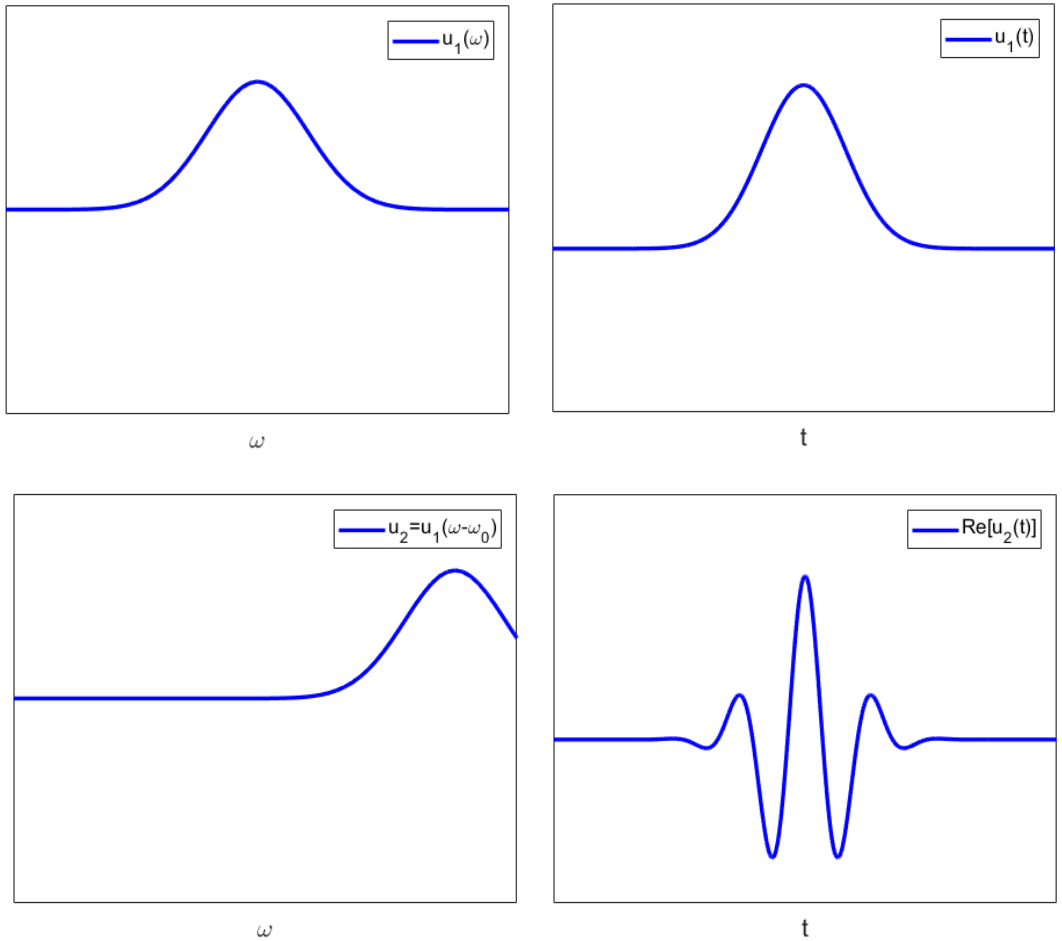


Figure 4.4. *The effect of a frequency shift onto the time-domain signal.*

4.3 Properties of ID Fourier transform

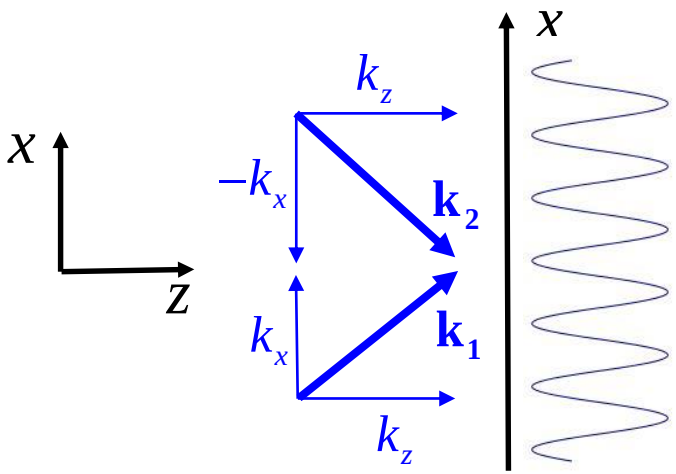
- Shift property:
  - Overlapping two signal of shifted spectra generates a sinusoidal temporal modulation:

$$\left|u(t)+u(t)e^{iw_0 t}\right| = 2\left|u(t)\right|^2 (1+\cos w_0 t).$$

  - In the *spatial* domain (off-axis plane waves at $z=z_0$):

$$I(\mathbf{r}) = \left|e^{i(-b_x x+b_z z_0)} + e^{i(bk_x x+b_z z_0)}\right|^2 =$$

$$= 2[1+\cos(2b_x x)]$$



$$I(x) \propto 1+\cos(2k_x x)$$

Figure 4.5. *The effect of a spatial frequency shift ($2k_x$) yields a modulation in the spatial domain (x).*

4.3 Properties of ID Fourier transform

- c) Parsevall's (Rayleigh's) theorem:

$$\int_{-\infty}^{\infty} \left| f(t) \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \tilde{f}(\omega) \right|^2 d\omega$$

$$\int_{-\infty}^{\infty} \left| f(x) \right|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \tilde{f}(k_x) \right|^2 dk_x \ .$$

- $I = \left\langle \left| f(t) \right|^2 \right\rangle$ the average intensity of the signal
- $S = \left| f(w) \right|^2$ the power spectrum of the signal

4.3 Properties of ID Fourier transform

- d) Similarity theorem:

$$f(at) \rightarrow \frac{1}{|a|} \cdot \tilde{f}\left(\frac{\omega}{a}\right)$$

$$f(ax) \rightarrow \frac{1}{|a|} \cdot \tilde{f}\left(\frac{k_x}{a}\right)$$

- The narrower the function, the broader its Fourier transform and vice-versa.

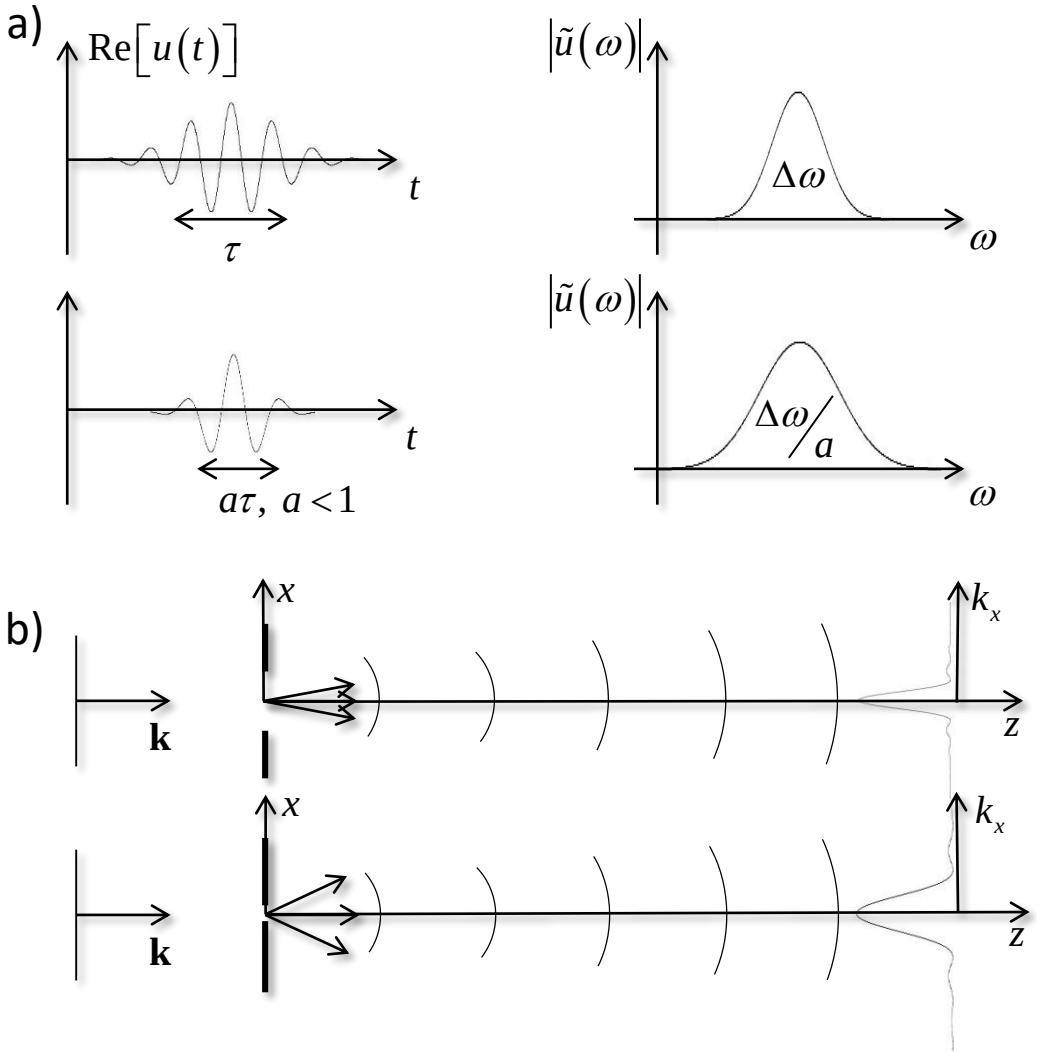


Figure 4.6. *The similarity theorem in the 1D temporal (a) and spatial domain (b).*

4.3 Properties of ID Fourier transform

- e) Convolution theorem:
  - The convolution operation of two functions *f* and *g* is defined as:

$$f \circledV_t g = \int_{-\infty}^{\infty} f(t')\,g(t-t')\,dt'$$

$$f \circledV_x g = \int_{-\infty}^{\infty} f(x')\,g(x-x')\,dx'.$$

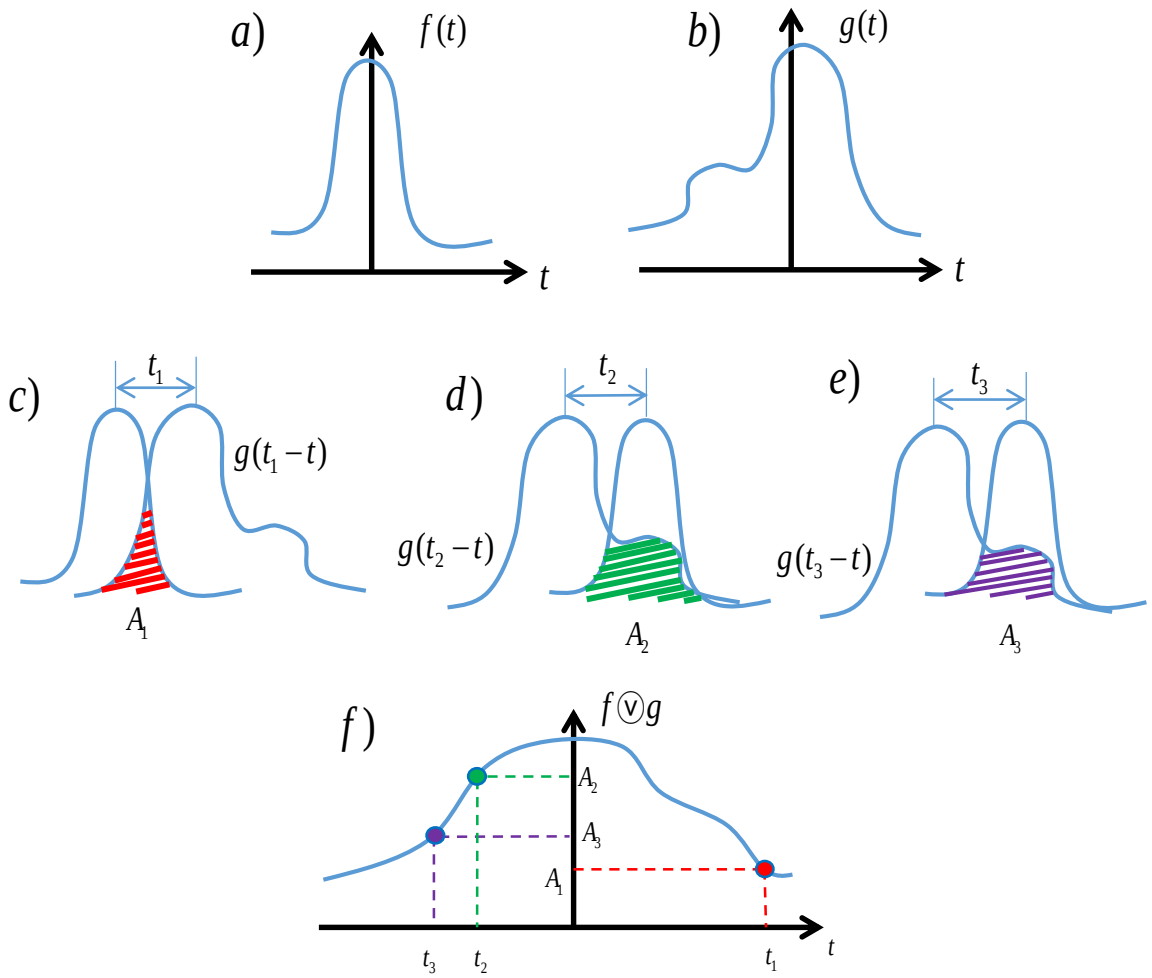


Figure 4.7. *Graphical illustration of the convolution operation.*

4.3 Properties of ID Fourier transform

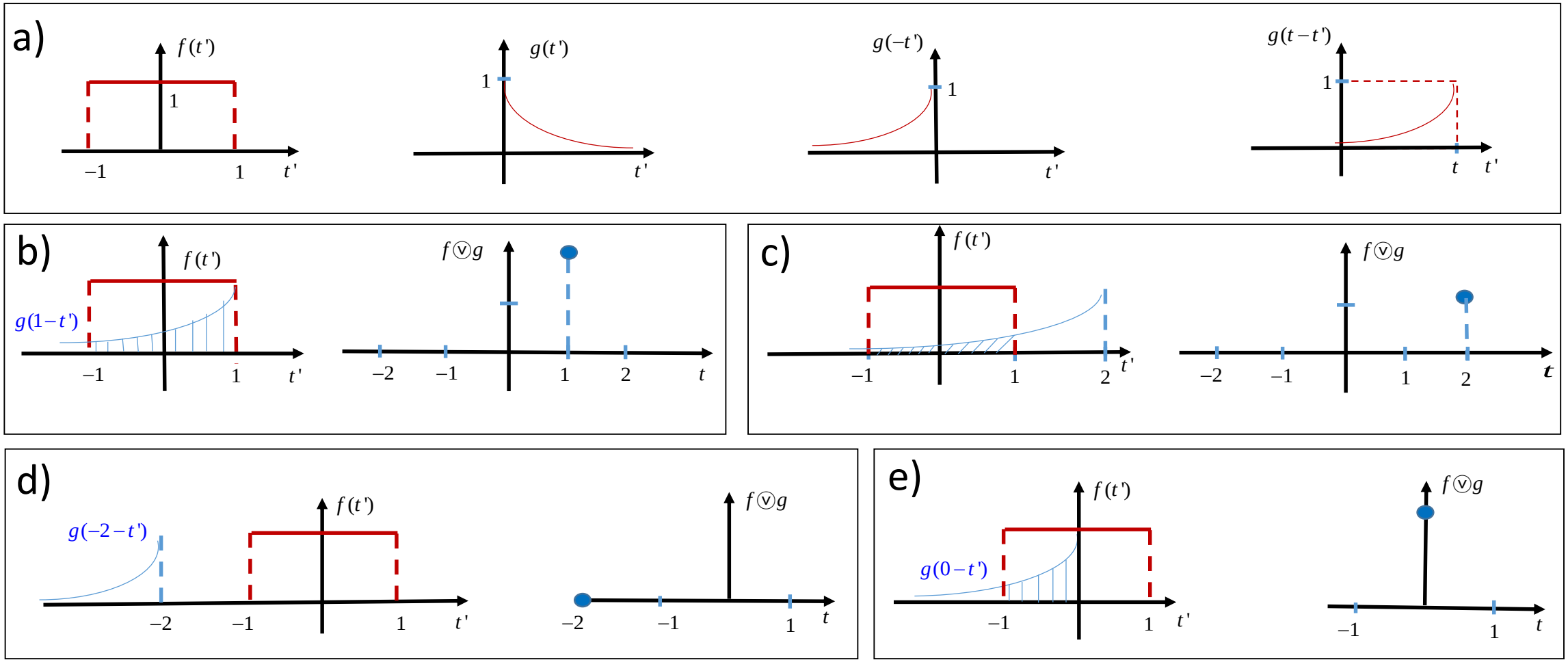


Figure 4.8. *Convolution between a rectangular function, f, and a single-sided exponential.*

4.3 Properties of ID Fourier transform

- Convolution theorem:
  - In the frequency domain, convolution operation becomes a product:

$$f(t) \circledast_t g(t) \leftrightarrow f(\omega)\, g(\omega)$$

  - Commutative:

$$f \circledast g = g \circledast f$$

  - Associative:

$$f \circledast (g \circledast h) = (f \circledast g) \circledast h$$

  - Distributive:

$$f \circledast (g + h) = f \circledast g + f \circledast h.$$

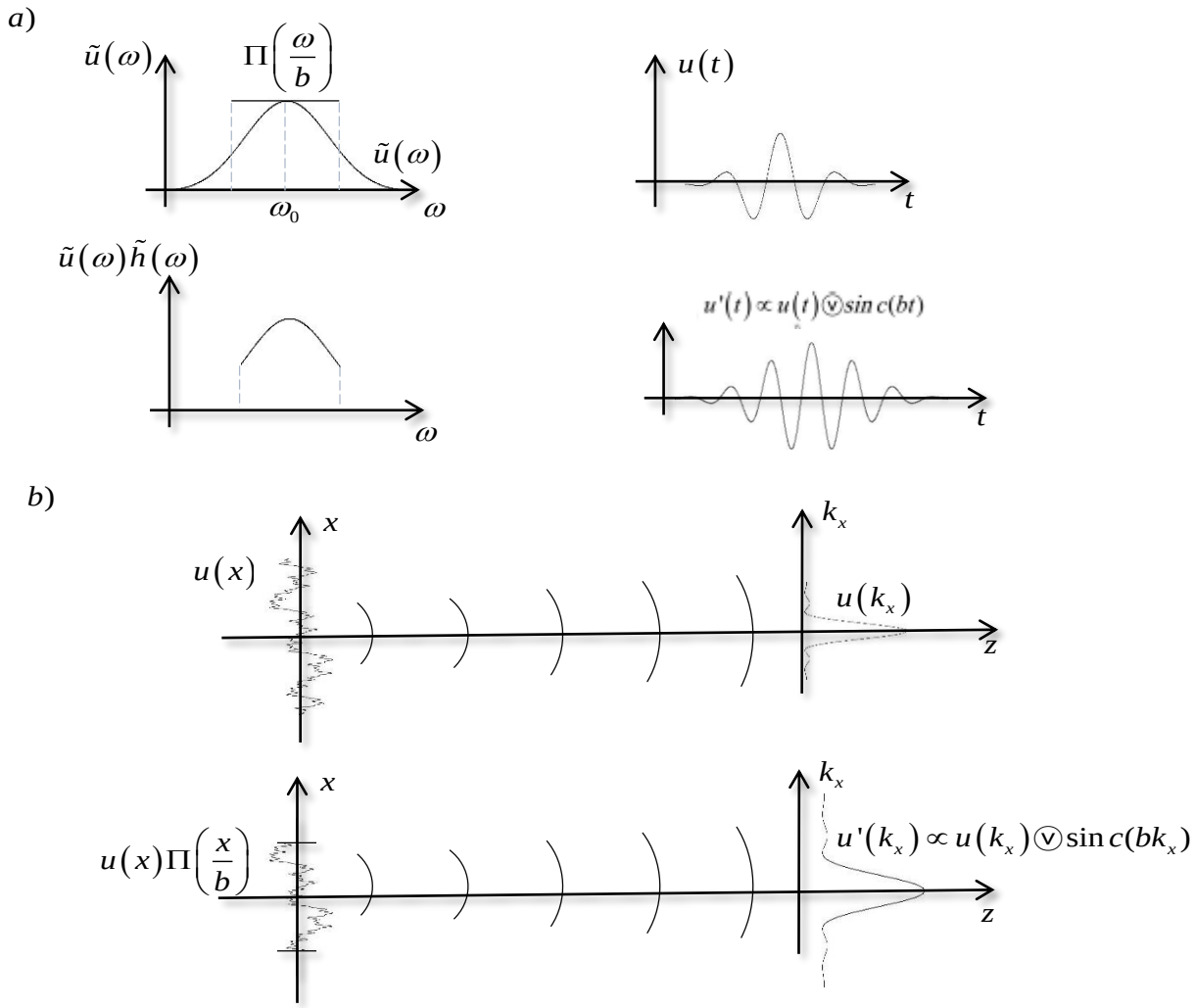


Figure 4.9. *Graphical illustration of the convolution operation.*

4.3 Properties of ID Fourier transform

- Transfer function of a linear system:
  - Output g is the convolution between the input f and the impulse response h:

$$g(t) = f \circledast h\,.$$

$$\tilde{g}(\omega) = \tilde{f}(\omega)\tilde{h}(\omega)$$

  - $\tilde{h}(\omega)$, transfer function

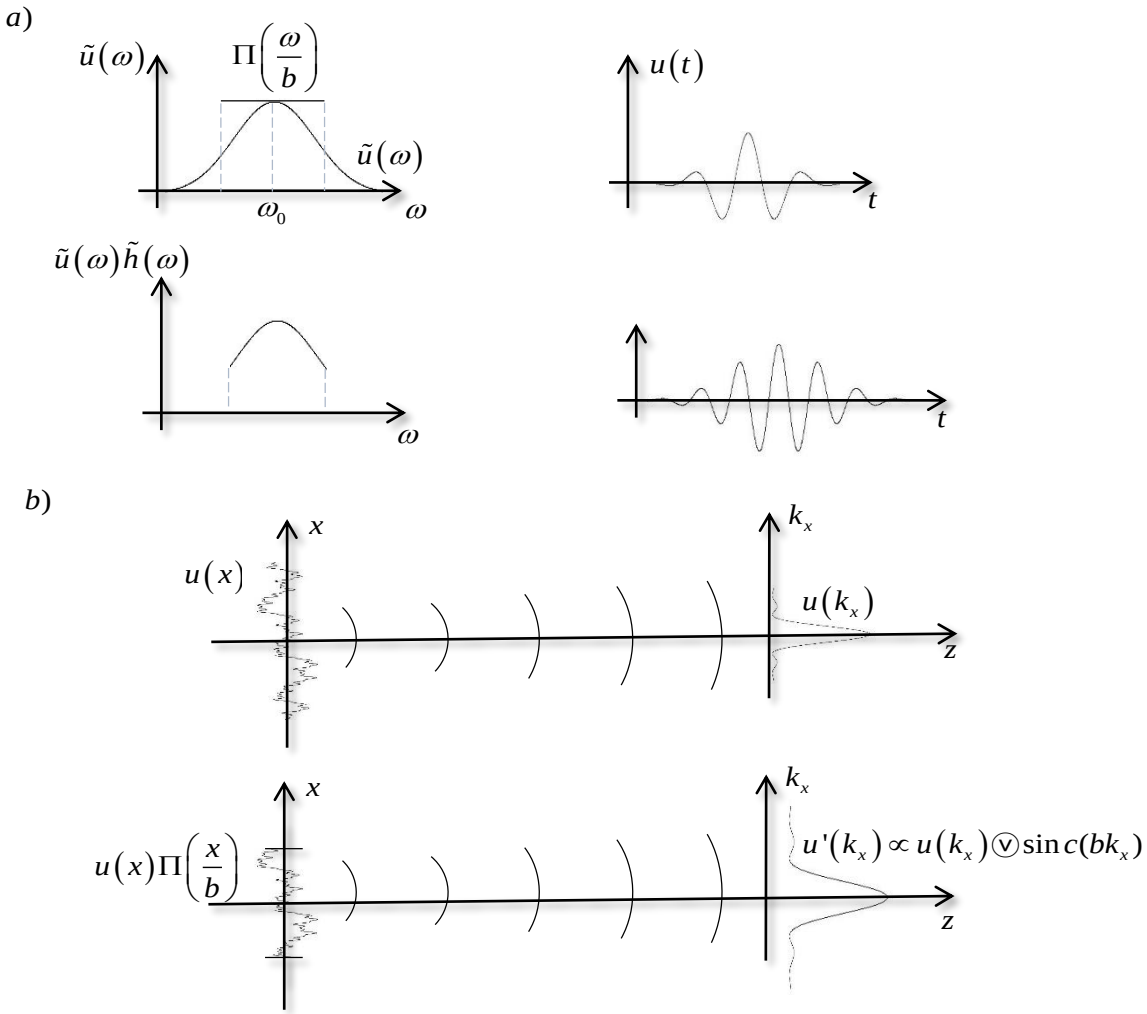


Figure 4.9. *a) Truncating the Fourier transform of a signal. b) Truncating a signal in the spatial domain*

4.3 Properties of ID Fourier transform

- g) Correlation theorem:

$$f \otimes_x g = \int_{-\infty}^{\infty} f(x')g(x'-x)dx'$$

  - g is not flipped.

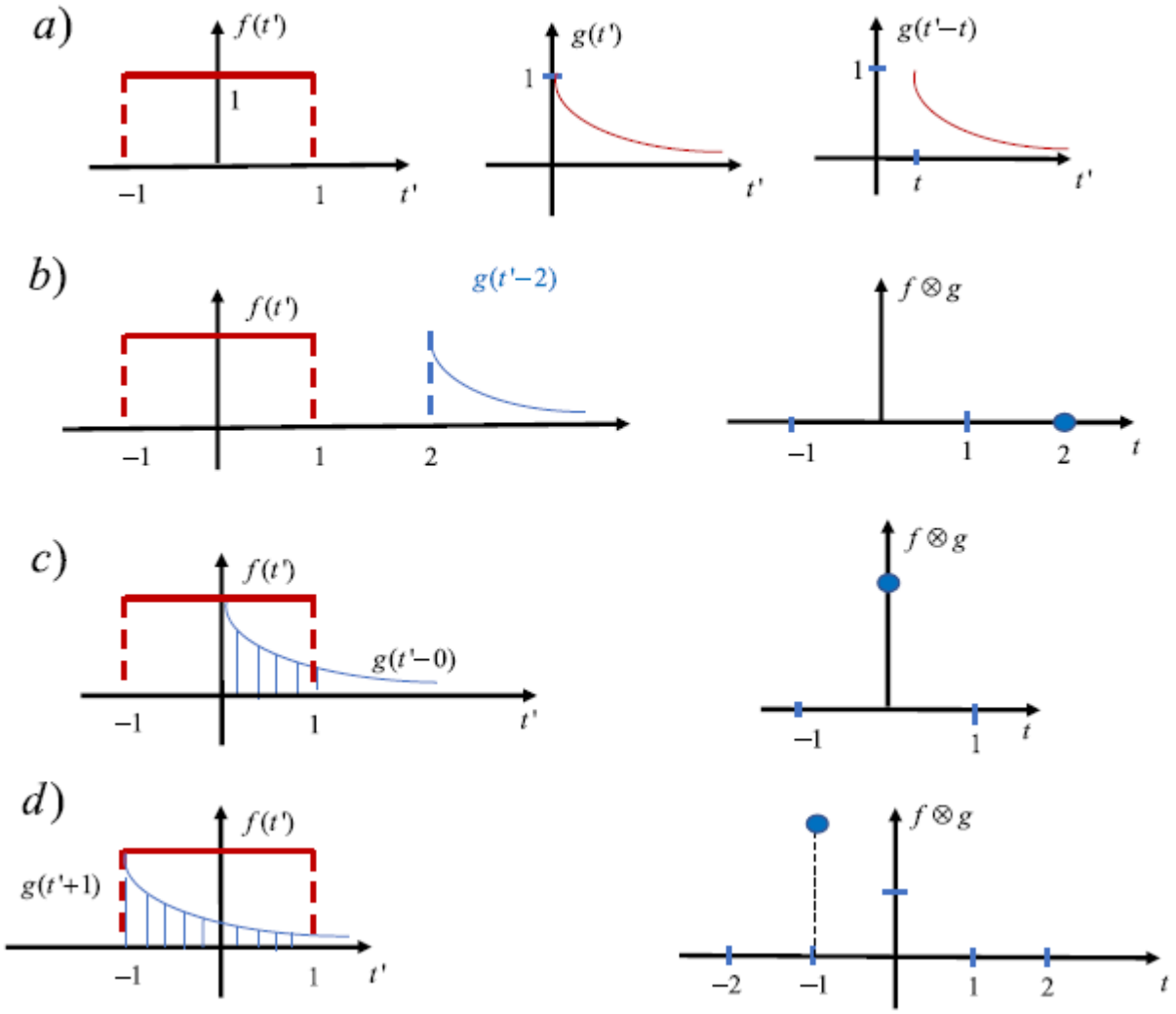


Figure 4.10. *Correlation procedure.*

4.3 Properties of ID Fourier transform

- Correlation theorem:
  - In the frequency domain, the correlation function also becomes a product:

$$f \otimes_x g \rightarrow \tilde{f}(k_x)\tilde{g}^*(k_x)$$

$$\tilde{f}(k_x) \otimes_{k_x} \tilde{g}(k_x) \rightarrow f(x) g^*(x)$$

  - *Autocorrelation:*

$$f \otimes_t f \leftrightarrow \tilde{f}(\omega)\tilde{f}^*(\omega) = \left|\tilde{f}(\omega)\right|^2 \triangleq S(\omega)$$

$$\text{if } g(x) = g(-x) \text{ then } f \circledv g = f \otimes g.$$

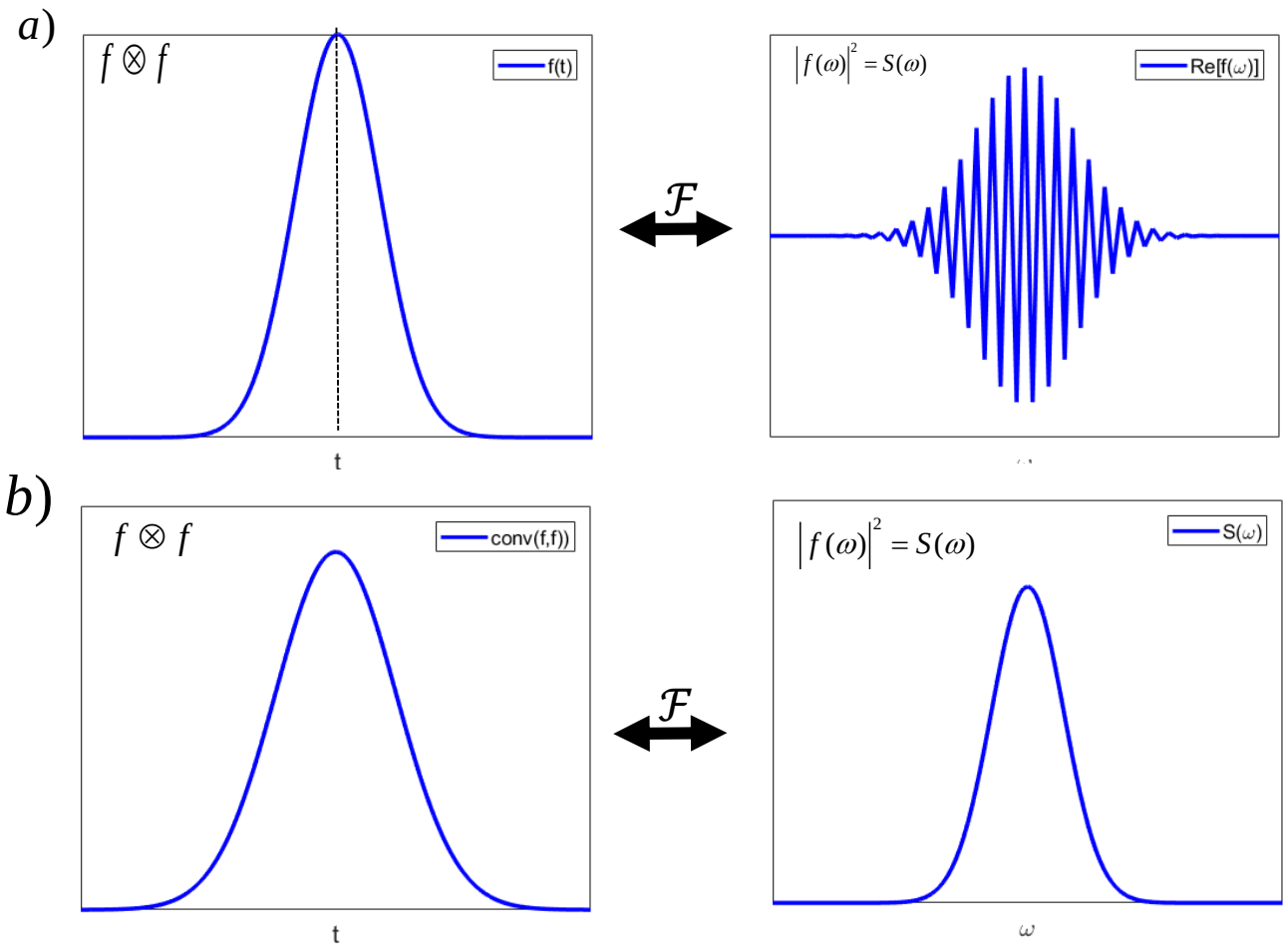


Figure 4.11. *Correlation procedure. Power spectrum of a signal can be obtained by Fourier transforming the autocorrelation function (and vice-versa).*

4.3 Properties of ID Fourier transform

- h) Differentiation theorem:

$$\frac{\partial f(t)}{\partial t} \leftrightarrow -i\omega \tilde{f}(\omega)$$

  - n^{th} order derivative and its Fourier transform:

$$\frac{\partial^n f(t)}{\partial t^n} \leftrightarrow (-i\omega)^n \tilde{f}(\omega)$$

  - Autocorrelation of the derivative of a function is the second order derivative of the autocorrelation:

$$\frac{\partial f(t)}{\partial t} \otimes \frac{\partial f(t)}{\partial t} = \frac{-\partial^2 [f \otimes f]}{\partial t^2}.$$

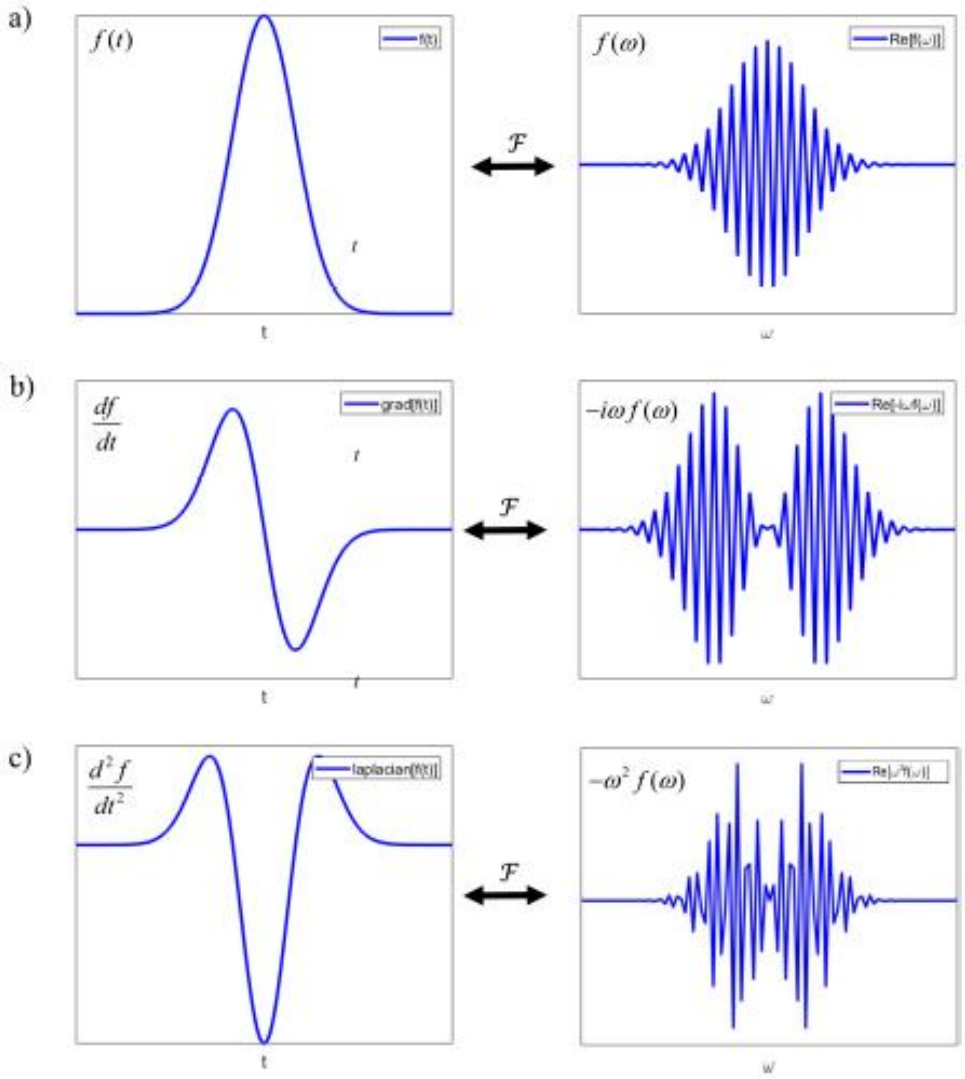


Figure 4.12. *Differentiation theorem.*

4.3 Properties of ID Fourier transform

- i) Moment theorem:
  - The nth moment, m_n, of a function $f(t)$ is defined as:

$$m_n = \int_{-\infty}^{\infty} t^n f(t)dt, n = 0,1,2, \ldots$$

  - Differentiation property: $t^n f(t) \leftrightarrow (-1)^n \frac{\partial^n \tilde{f}(\omega)}{\partial \omega^n}$.
  - Central ordinate theorem:

$$m_n = \int_{-\infty}^{\infty} t^n f(t)dt = (-i)^n \left. \frac{\partial^n \tilde{f}(\omega)}{\partial \omega^n} \right|_{\omega=0}.$$

  - Frequency-domain moments:

$$\int_{-\infty}^{\infty} \omega^n \tilde{f}(\omega)d\omega = i^n \left. \frac{\partial^n f(t)}{\partial t^n} \right|_{t=0}$$

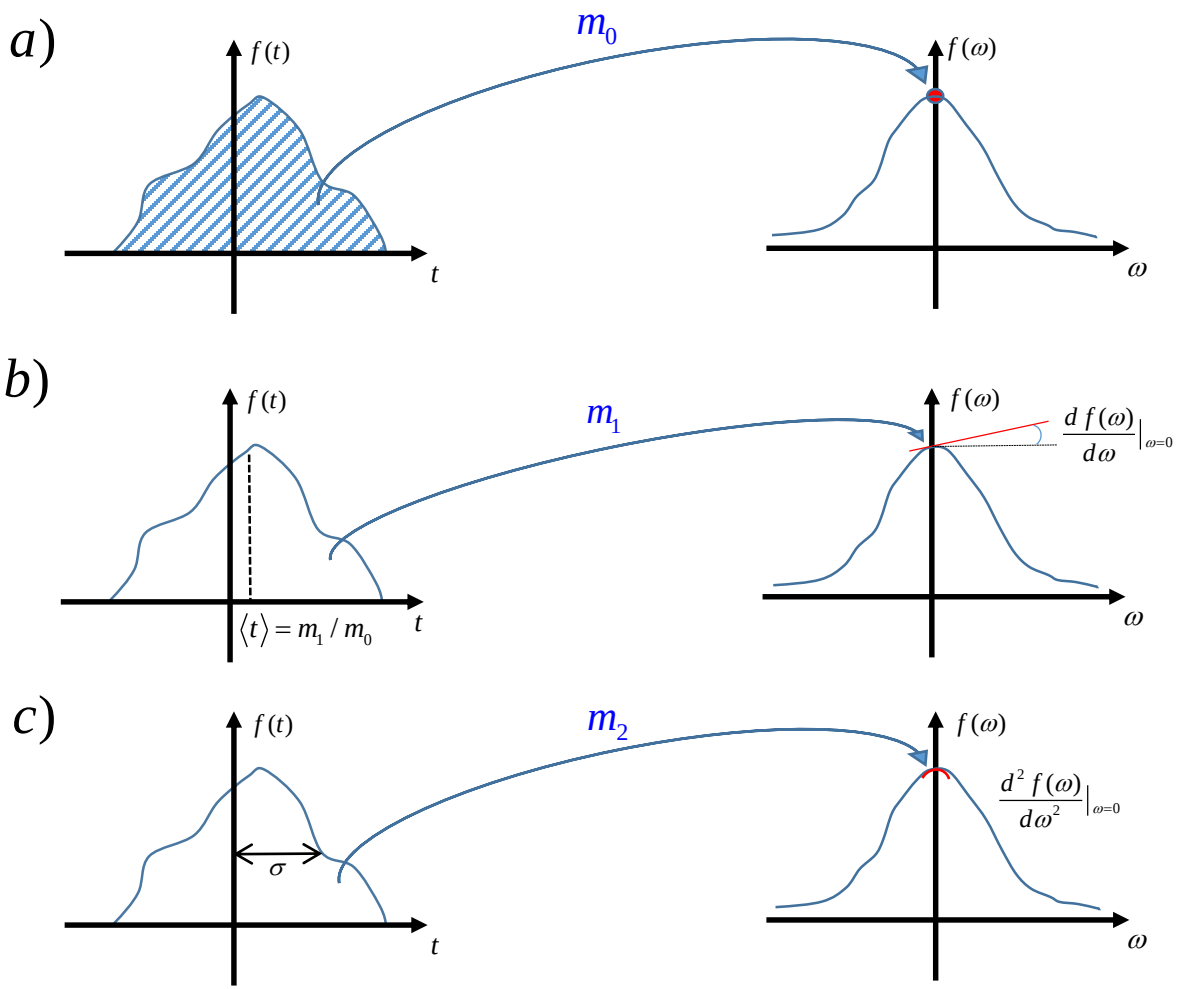


Figure 4.13. *Moment theorems.*

- Mean

$$\langle x \rangle = \int_{-\infty}^{\infty} x f(x) dx = m_1 = i \frac{\partial \tilde{f}(k_x)}{\partial k_x} \bigg|_{k_x=0}$$

- Variance

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} (x - \langle x \rangle)^2 f(x) dx \\
&= \int_{-\infty}^{\infty} x^2 f(x) dx - 2\langle x \rangle \int_{-\infty}^{\infty} x f(x) dx + \langle x \rangle^2 \int_{-\infty}^{\infty} f(x) dx \\
&= \langle x^2 \rangle - \langle x \rangle^2 \\
&= m_2 - m_1{}^2 \\
&= - \frac{\partial^2 \tilde{f}(k_x)}{\partial k_x{}^2} \bigg|_{k_x=0} - \left[i \frac{\partial \tilde{f}(k_x)}{\partial k_x} \bigg|_{k_x=0} \right]^2 .
\end{aligned}$$

Note the sign change, *t vs. x*

$$\frac{\partial^n f(t)}{\partial t^n} \leftrightarrow (-i\omega)^n \tilde{f}(\omega) \qquad a)$$

$$\frac{\partial^n \tilde{f}(\omega)}{\partial \omega^n} \leftrightarrow (it)^n f(t) \qquad b)$$

$$\frac{\partial^n f(x)}{\partial x^n} \leftrightarrow (ik_x)^n \tilde{f}(k_x) \qquad c)$$

$$\frac{\partial^n \tilde{f}(k_x)}{\partial k_x{}^n} \leftrightarrow (-ix)^n f(x) \qquad d) .$$

4.3 Properties of ID Fourier transform

- j) Moments of a convolution:

$$h = f \circledast g.$$

$$\int_{-\infty}^{\infty} h(x)\,dx = \tilde{h}(k_x)\Big|_{k_x=0}$$

$$= \tilde{f}(k_x)\,\tilde{g}(k_x)\Big|_{k_x=0}$$

$$= \left(\int_{-\infty}^{\infty} f(x)\,dx\right)\left(\int_{-\infty}^{\infty} g(x)\,dx\right).$$

-Zeroth order moment is product of the two (are is product of the two areas)

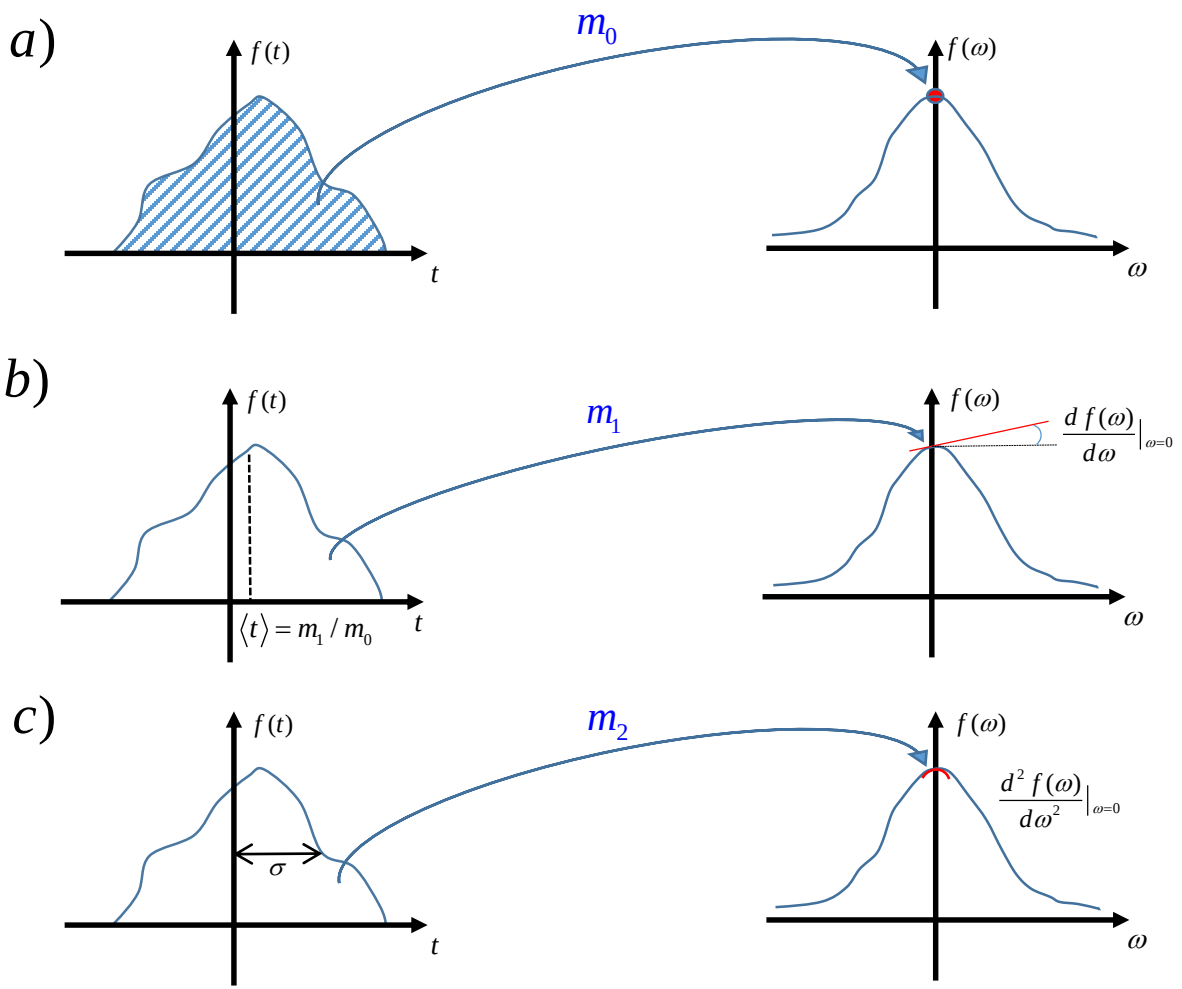


Figure 4.13. *Moment theorems.*

- Mean (1st order moment)

$$\langle x\rangle_h = \frac{\int_{-\infty}^{\infty} x h(x)dx}{\int_{-\infty}^{\infty} h(x)dx} = \frac{i\frac{\partial}{\partial k_x}[\tilde{f}(k_x)\tilde{g}(k_x)]|_{k_x=0}}{[\tilde{f}(k_x)\tilde{g}(k_x)]|_{k_x=0}}$$

$$= \frac{\left[i\frac{\partial \tilde{f}(k_x)}{\partial k_x}\cdot \tilde{g}(k_x)|_{k_x=0} + i\tilde{f}(k_x)\frac{\partial \tilde{g}(k_x)}{\partial k_x}|_{k_x=0}\right]}{[\tilde{f}(k_x)\tilde{g}(k_x)]|_{k_x=0}}$$

$$= \langle x\rangle_f + \langle x\rangle_g .$$

- Second-order moment

$$\langle x^2\rangle_h = \langle x^2\rangle_f + \langle x^2\rangle_g + 2\langle x\rangle_f\langle x\rangle_g$$

- If *f* and *g* are zero-average, the variance of the convolution is

$$\sigma_h{}^2 = \sigma_f{}^2 + \sigma_g{}^2$$

4.4 Common 1D Fourier transform pairs

- Rectangular function, Π :

$$\mathrm{P}(x) = \begin{cases} 1, & x \,\hat{\mathrm{I}}\, \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \\ \tfrac{1}{2}, & |x| = \tfrac{1}{2} \\ 0, & \text{rest} \end{cases} .$$

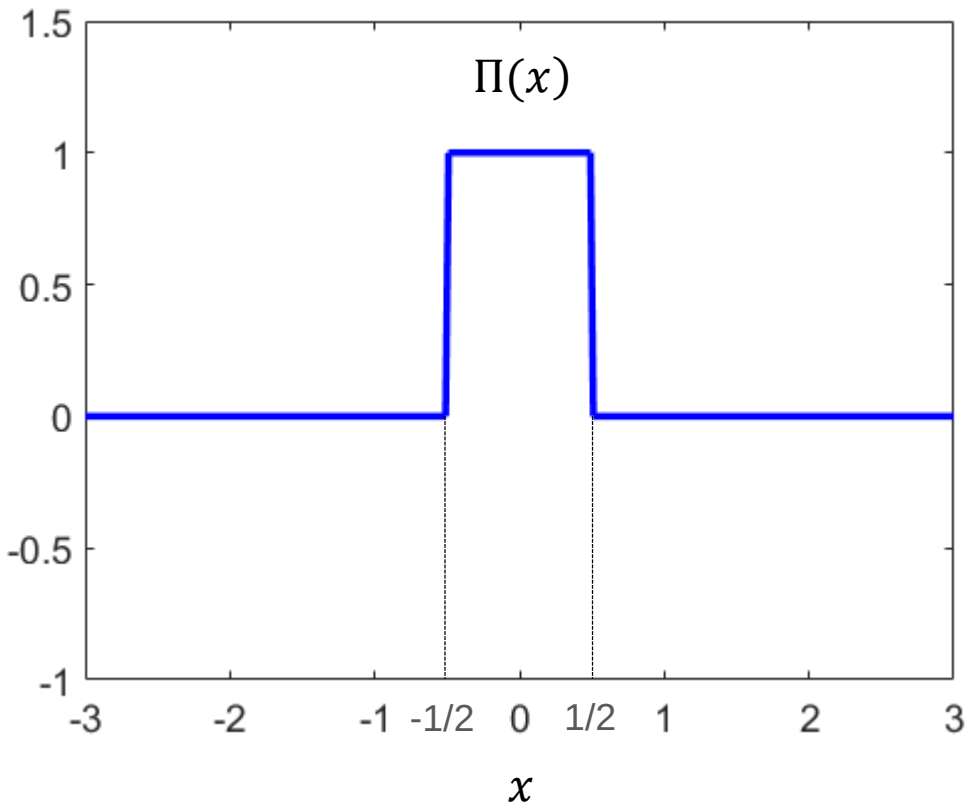


Figure 4.14. *Rectangular function, Π.*

4.4 Common 1D Fourier transform pairs

b) Triangle function, Λ

$$\Lambda(x) = \begin{cases} 1-|x|, & x \in (-1,1) \\ 0, & \text{otherwise} \end{cases}$$

Symbol – Λ : Capital Lambda (looks like a triangle function)

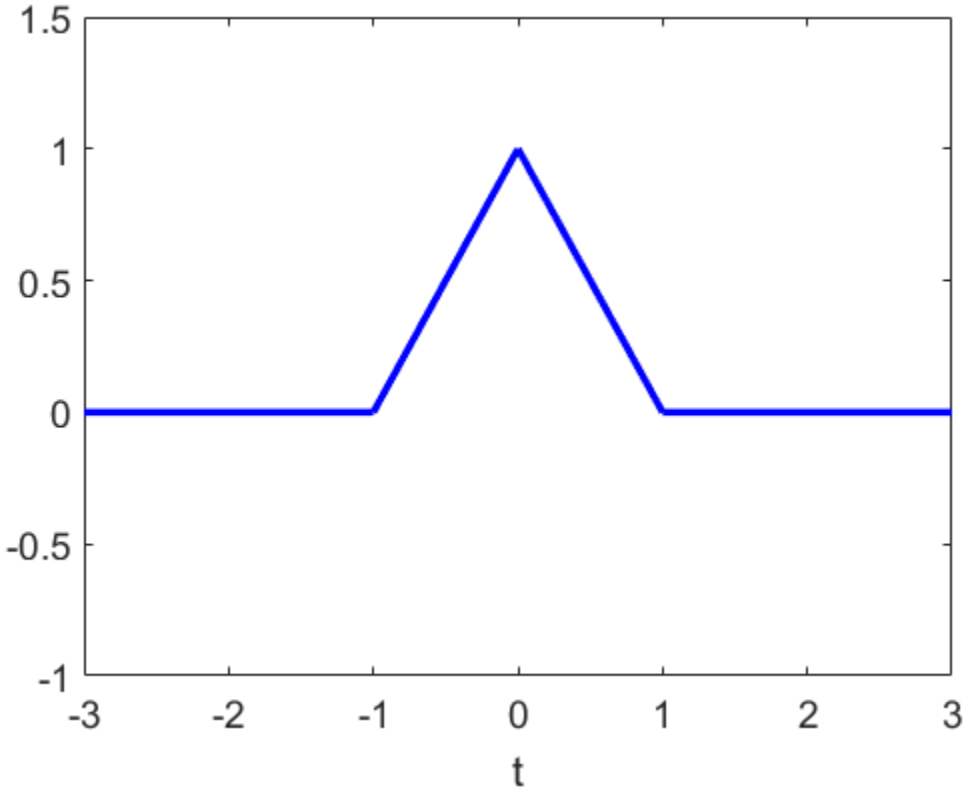


Figure 4.15. *Triangle function, Λ*

4.4 Common 1D Fourier transform pairs

c) Heaviside or step, Γ

$$\Gamma(x) = \begin{cases} 1, & x > 0 \\ 1/2, & x = 0 \\ 0, & x < 0 \end{cases}$$

Symbol – Γ : Capital Gamma (looks like a step function)

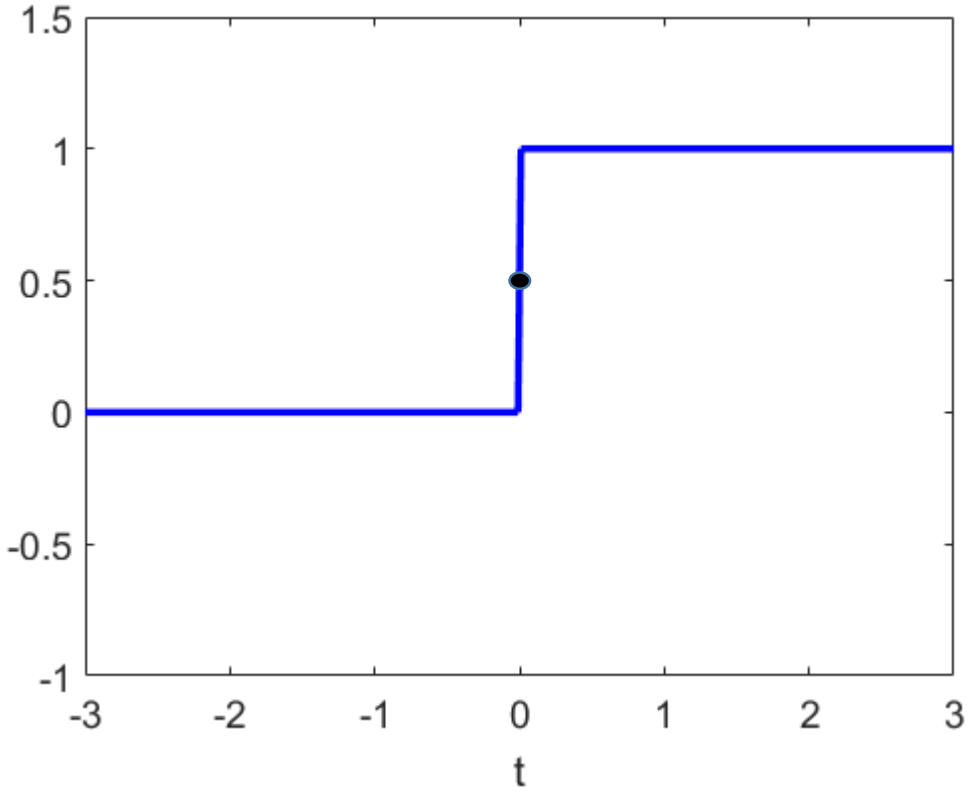


Figure 4.16. *Step function, Γ*

4.4 Common 1D Fourier transform pairs

d) Signum, or sign function

$$\operatorname{sign}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Note – $\Gamma(x) = [1 + sign(x)]/2$

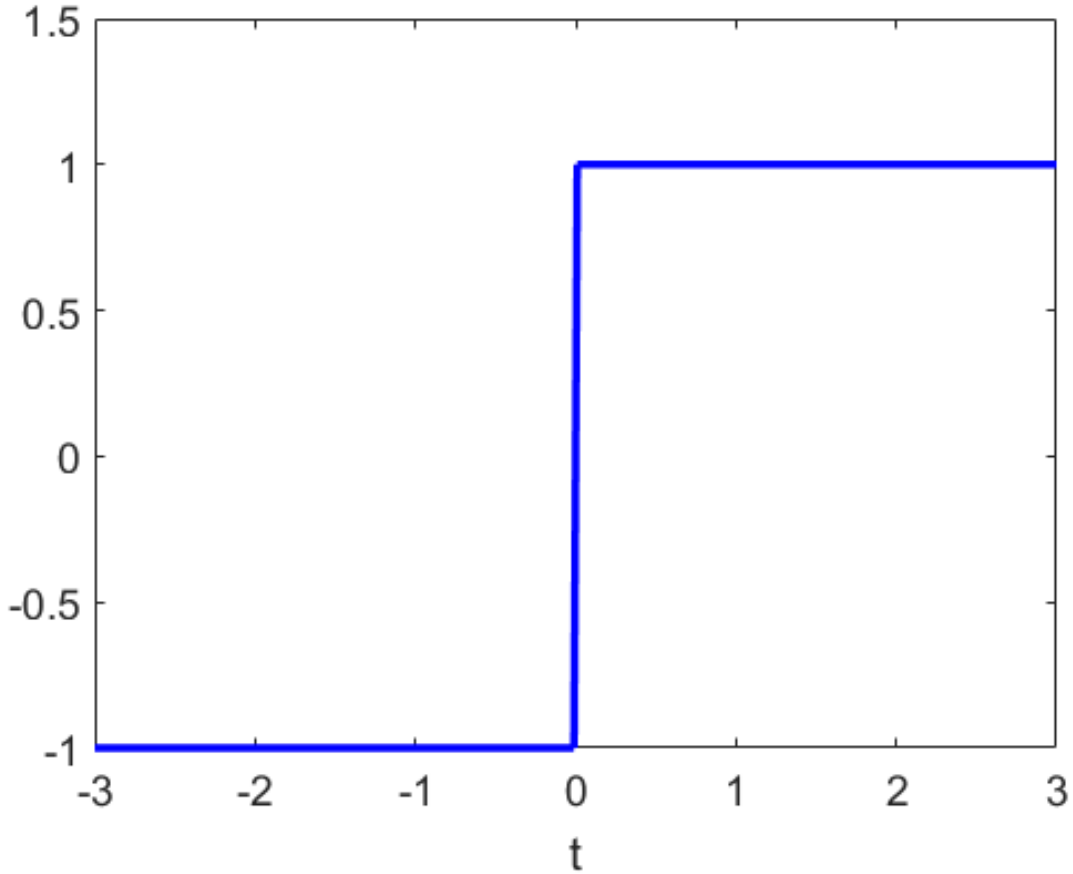


Figure 4.17. *Signum function*

4.4 Common 1D Fourier transform pairs

e) Comb function

$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

Called comb because it looks like one!

Recall – $\delta(\frac{x-a}{b}) = |b|\,\delta(x-a)$

Thus, $comb(\frac{x-a}{b}) = |b| \sum_{n=-\infty}^{\infty} \delta(x-a-nb)$

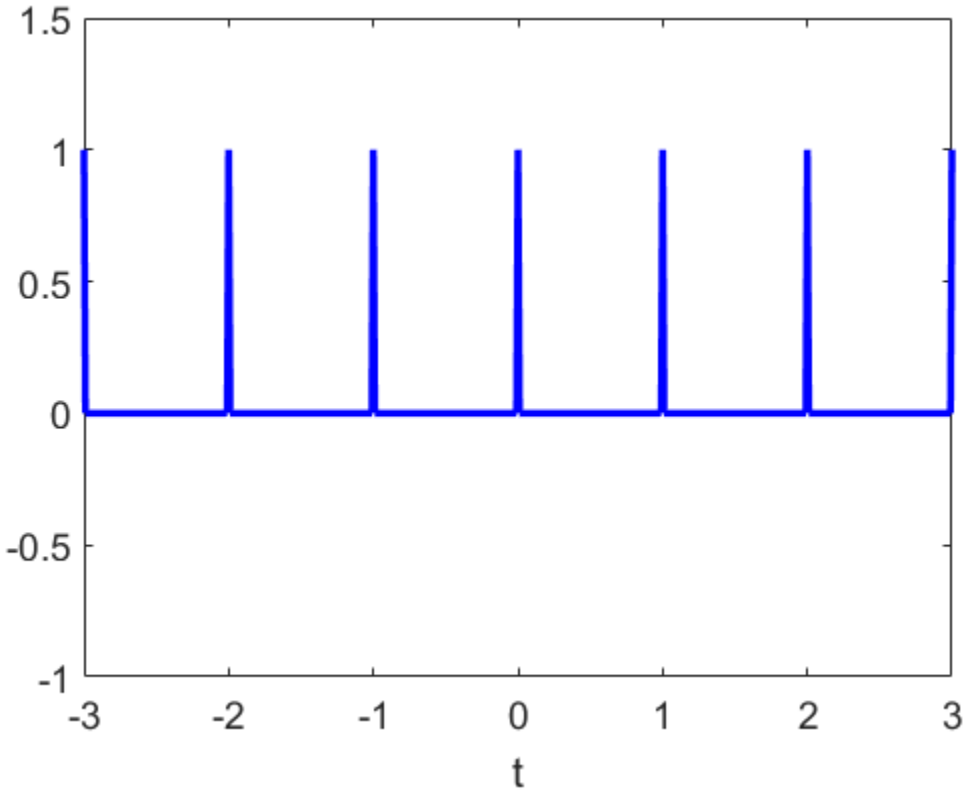


Figure 4.18. *Comb function*

4.4 Common 1D Fourier transform pairs

f) Sinc function

$$\operatorname{sinc}(x) = \frac{\sin x/2}{x/2}$$

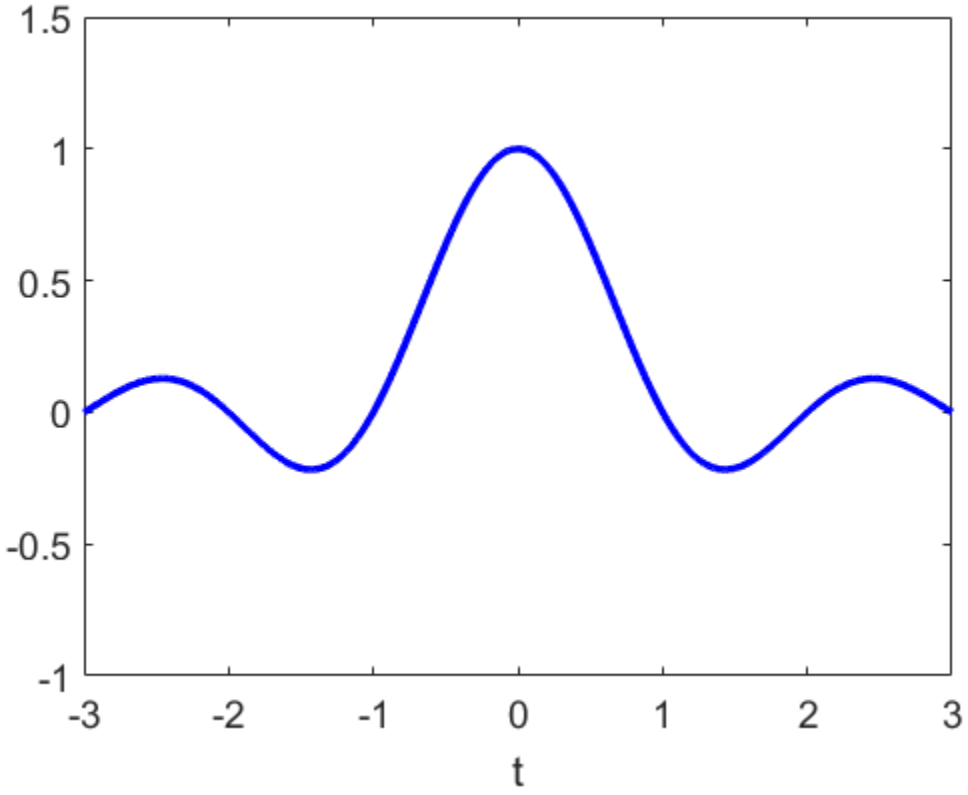


Figure 4.19. *Sinc function*

4.4 Common 1D Fourier transform pairs

g) Gaussian function

$$e^{-\frac{x^2}{2}}$$

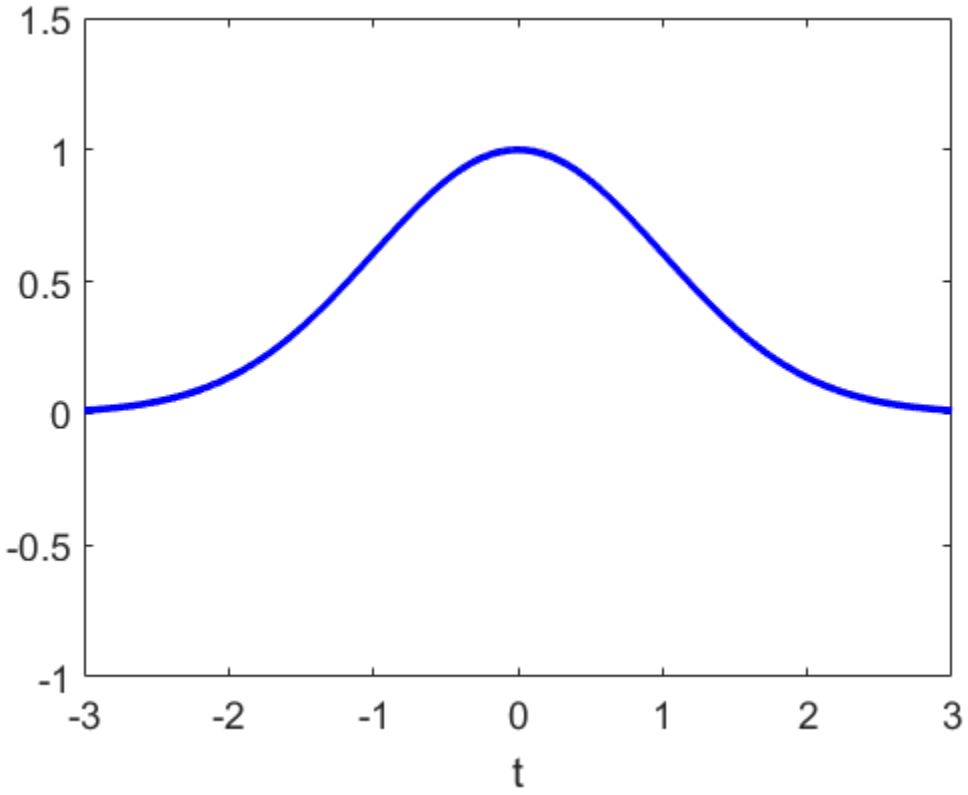


Figure 4.20. *Gaussian function*

4.4 Common 1D Fourier transform pairs

h) Lorentzian function

$$\frac{1}{1+x^2}$$

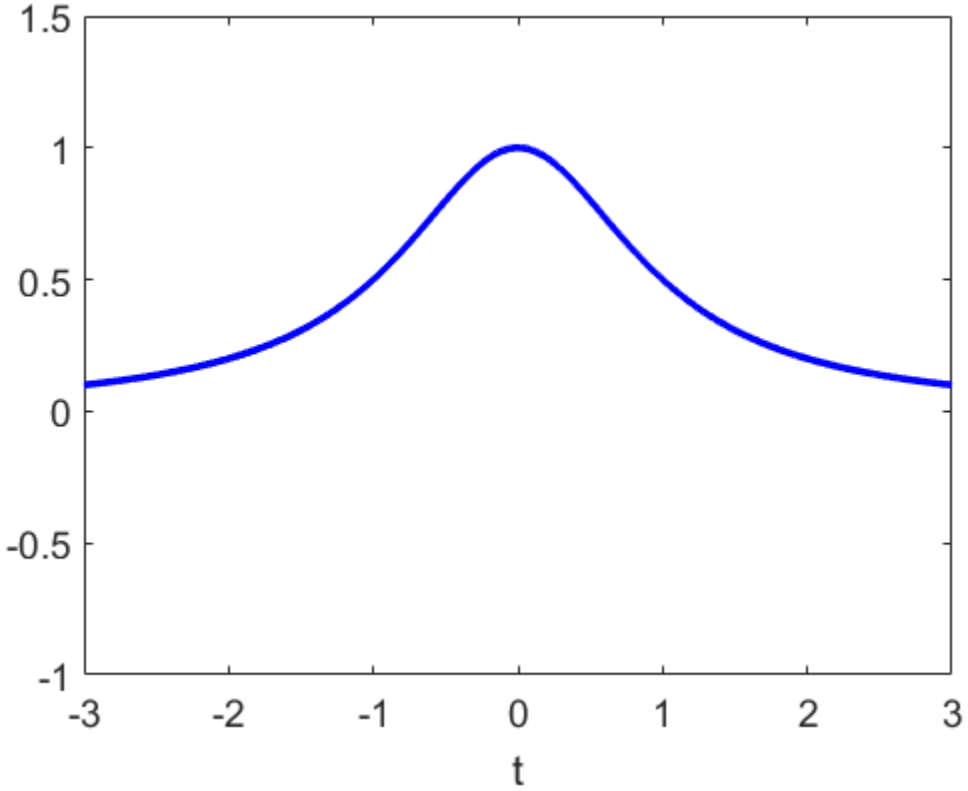


Figure 4.21. *Lorentzian function*

4.4 Common 1D Fourier transform pairs

Commonly Encountered Fourier Transform Pairs

$f(t)$	$\tilde{f}(\omega)$
1	$\delta(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{i\omega t_0}$
$e^{-i\omega_0 t}$	$\delta(\omega+\omega_0)$
$\Pi\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}(\omega T)$
$\mathrm{sinc}\left(\frac{t}{T}\right)$	$T\Pi(\omega T)$
$\Lambda\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}^2(\omega T)$
$\mathrm{sinc}^2\left(\frac{t}{T}\right)$	$T\Lambda(\omega T)$

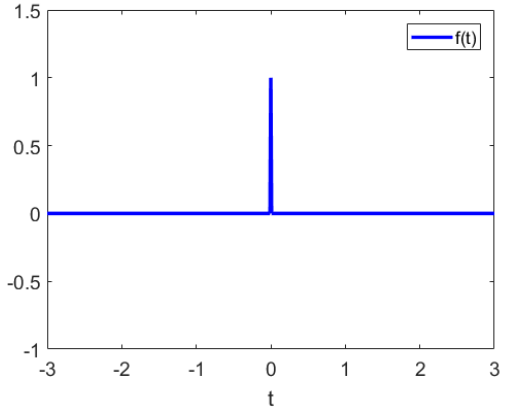
$f(t)$	$\tilde{f}(\omega)$
$e^{-\frac{t^2}{2b^2}}$	$be^{-\frac{b^2\omega^2}{2}}$
$\cos(\omega_0 t)$	$\frac{1}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin(\omega_0 t)$	$\frac{i}{2}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$
$\Gamma(t)$	$\pi\delta(\omega)+\frac{i}{\omega}$
$\mathrm{sign}(t)$	$\frac{i}{\omega}$
$e^{-\alpha\lvert t\rvert}$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$\Gamma(t)e^{-\alpha t}$	$\frac{1}{\alpha-i\omega}$

4.4 Common 1D Fourier transform pairs

Commonly Encountered Fourier Transform Pairs

$a)$

$\delta(t)$



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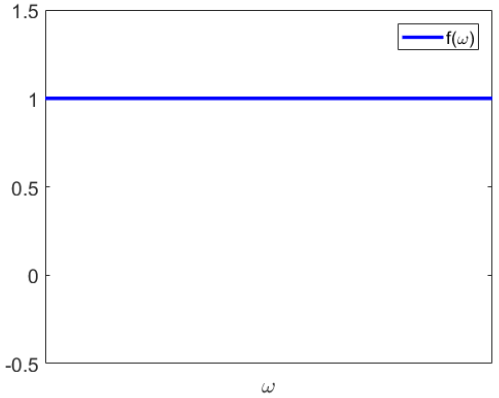
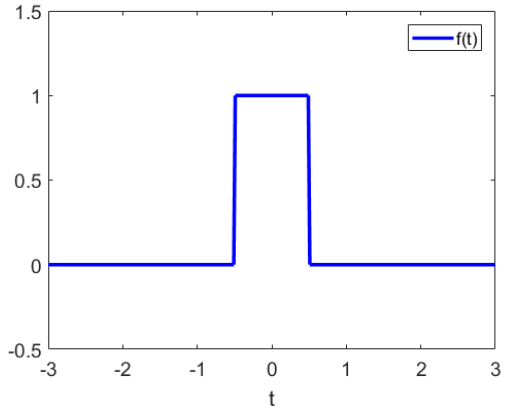
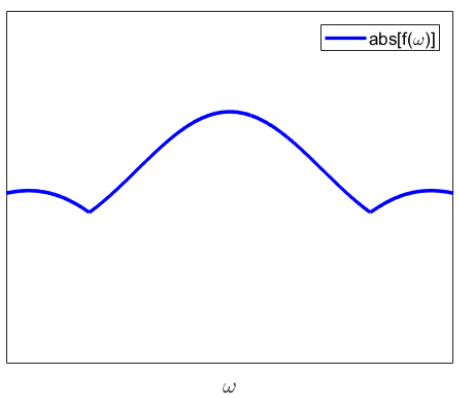


$c)$

$\Pi(\frac{t}{T})$

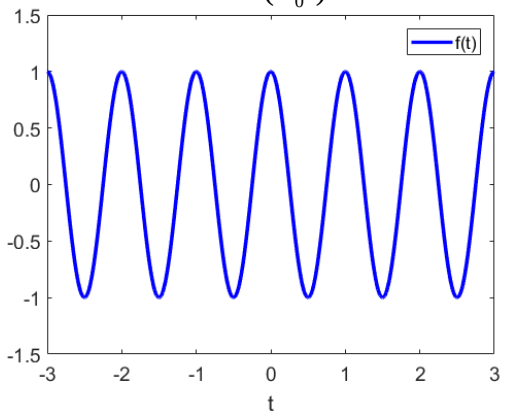


$T\mathrm{sinc}(\omega T)$

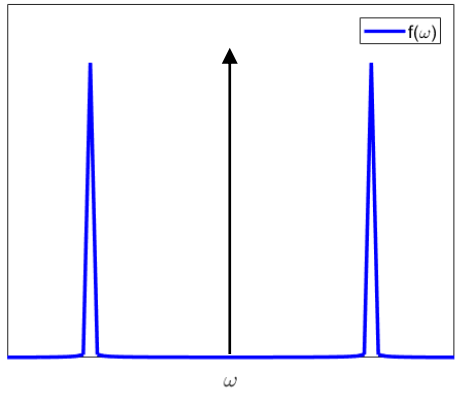


$b)$

$\cos(w_0 t)$

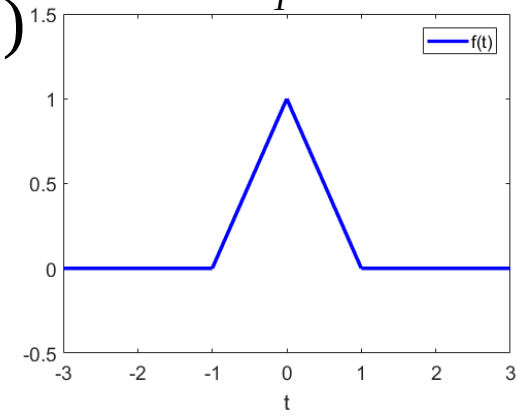


$[d(w-\ w_0)+d(w+\ w_0)]/2$

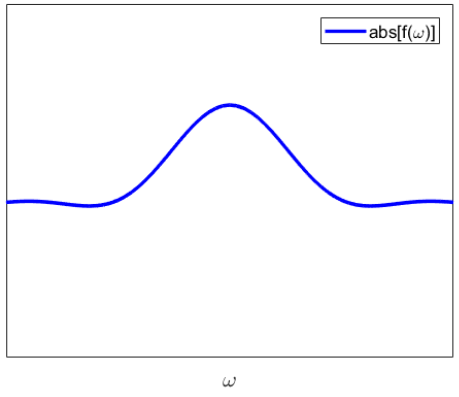


$d)$

$\Lambda(\frac{t}{T})$



$T\mathrm{sinc}^2(\omega T)$

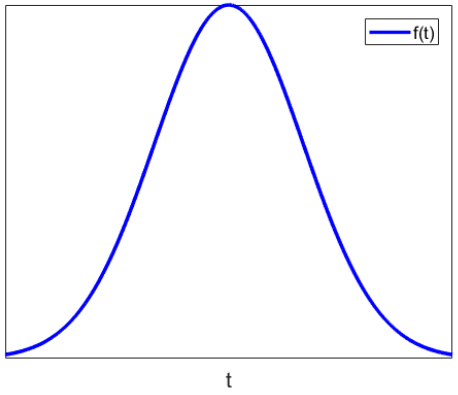


4.4 Common 1D Fourier transform pairs

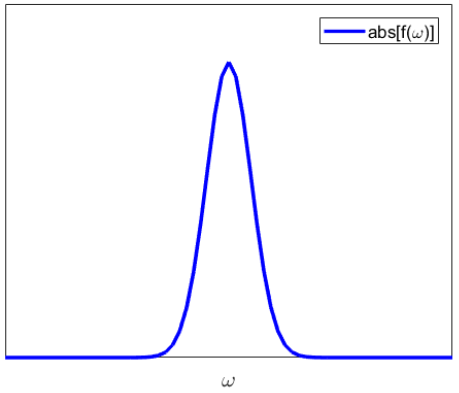
Commonly Encountered Fourier Transform Pairs

$e)$

$$e^{-\frac{t^2}{2b^2}}$$

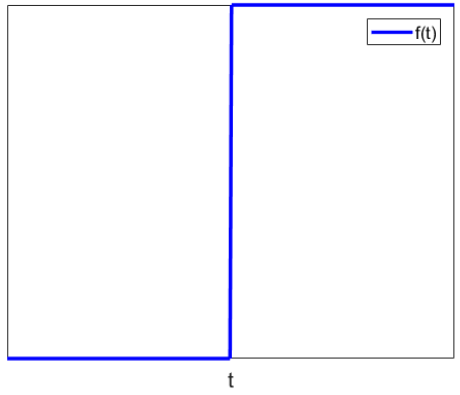


$$be^{-\frac{b^2\omega^2}{2}}$$

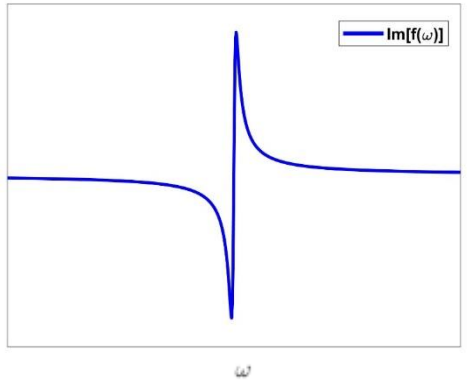


$f)$

$$\Gamma(t)e^{-\alpha t}$$



$$\frac{1}{\alpha - i\omega}$$



Note that we use functions of variables t and ω, but of course, these pairs apply equally well to the spatial domain.

Chapter 5 – 2D Fourier Transform

5.1 Definition

- To study imaging the concept of Fourier transforms must be generalized to 2D and 3D functions.
  - Diffraction and 2D image formation ⟶ 2D Fourier Transforms
  - Light scattering and tomographic reconstruction ⟶ 3D Fourier Transforms
- A 2D function *f* can be reconstructed from its Fourier components via an inverse Fourier transform

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(k_x,k_y)e^{i(k_x x + k_y y)}dk_x dk_y$$

- Conversely, the Fourier transform of *f* is

$$\tilde{f}(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(k_x x + k_y y)}dxdy$$

5.1 Definition

- Thus $\tilde{f}(k_x,k_y)$ assigns the amplitude and phase associated with the harmonic of frequency **k**= (k_x,k_y).
- For a fixed frequency (k_x,k_y), the spatial dependence of phase is given by

$$\phi(x,y) = \mathbf{k}\cdot\mathbf{r}$$
$$= k_x x + k_y y$$

$$\phi(x,y) = |k|\left(x\frac{k_x}{|k|} + y\frac{k_y}{|k|}\right) \quad \text{where} \quad |k| = \sqrt{k_x{}^2 + k_y{}^2}$$

- **k**- vector makes an angle $\theta = \tan^{-1}\left(\frac{k_y}{k_x}\right)$ with the x-axis
  - Thus, $\phi(x,y) = |k|\left(x\cos\theta + y\sin\theta\right)$
  - and wavelength (dist. over which the phase gains 2π along the direction of **k** is $\Lambda = \frac{2\pi}{|k|}$

5.1 Definition

- The wavefront is defined as the contour along which the phase is constant. The gradient of the phase is

$$\nabla\phi(x,y) = |k|\left(\frac{k_x}{|k|},\frac{k_y}{|k|}\right)$$
$$= \mathbf{k}$$

- Thus, the wavefront, along which the phase change vanishes is perpendicular to the **k**-vector, while the largest phase change takes place along the direction of **k.**

  - Note- the gradient of function *f(x,y,z)* is defined by the vector $\nabla f = \mathbf{x}\frac{\partial f}{\partial x} + \mathbf{y}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$

5.1 Definition

The Fourier transform indicates that the 2D function f is a superposition of waves with appropriate amplitude and phase for each frequency.

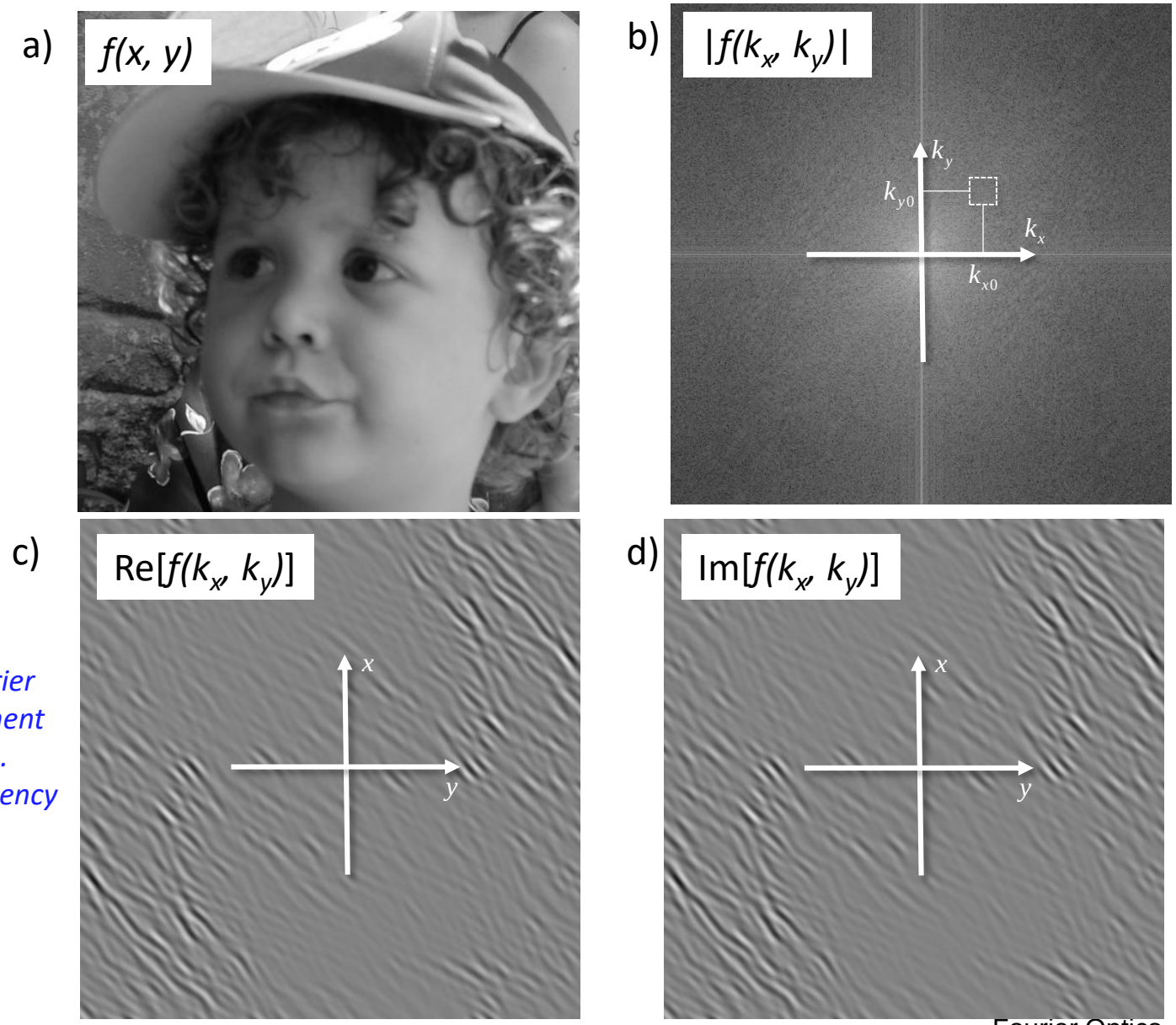


Figure 5.1. *a) Example of a 2D function f(x,y). b) The modulus of the Fourier transform (i.e. spectral amplitude). c) The real part of the Fourier component associated with the frequency (k_{x0},k_{y0}) indicated by the square region in b. d)The imaginary part of the Fourier component associated with the frequency (k_{x0},k_{y0}) indicated by the square region in b.*

5.1 Definition

- The convolution operation between two 2D functions *f(x,y)* and *g(x,y)* is defined as

$$f \circledv_{xy} g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')g(x-x',y-y')dx'dy'$$

- Sometimes, one can encounter one-dimensional convolutions of 2D functions,

$$f \circledv_{x} g = \int_{-\infty}^{\infty} f(x',y')g(x'-x,y')dx'$$

  - The symbol $\circledv_x$ indicates that the convolution operation is taken along *x* axis.

- Like in the 1D case, the only difference between convolution and correlation is in the sign of the argument of *g*.

5.1 Definition

- If a function f has separable variables and can be written in the form $f(x,y) = f_1(x) f_2(y)$ then the 2D Fourier transform factorizes into two 1D-Fourier transforms.

$$\tilde{f}(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-i(k_x x + k_y y)} dx dy$$

$$= \int_{-\infty}^{\infty} f_1(x) e^{-i(k_x x)} dx \int_{-\infty}^{\infty} f_2(y) e^{-i(k_y y)} dy$$

- Sometimes it is useful to express such a function using the following identity

$$f(x,y) = f_1(x) f_2(y)$$

$$= \left[f_1(x)\delta(y) \right] \circledv_{xy} \left[\delta(x) f_2(y) \right]$$

5.2 Significance of the Spectral Phase

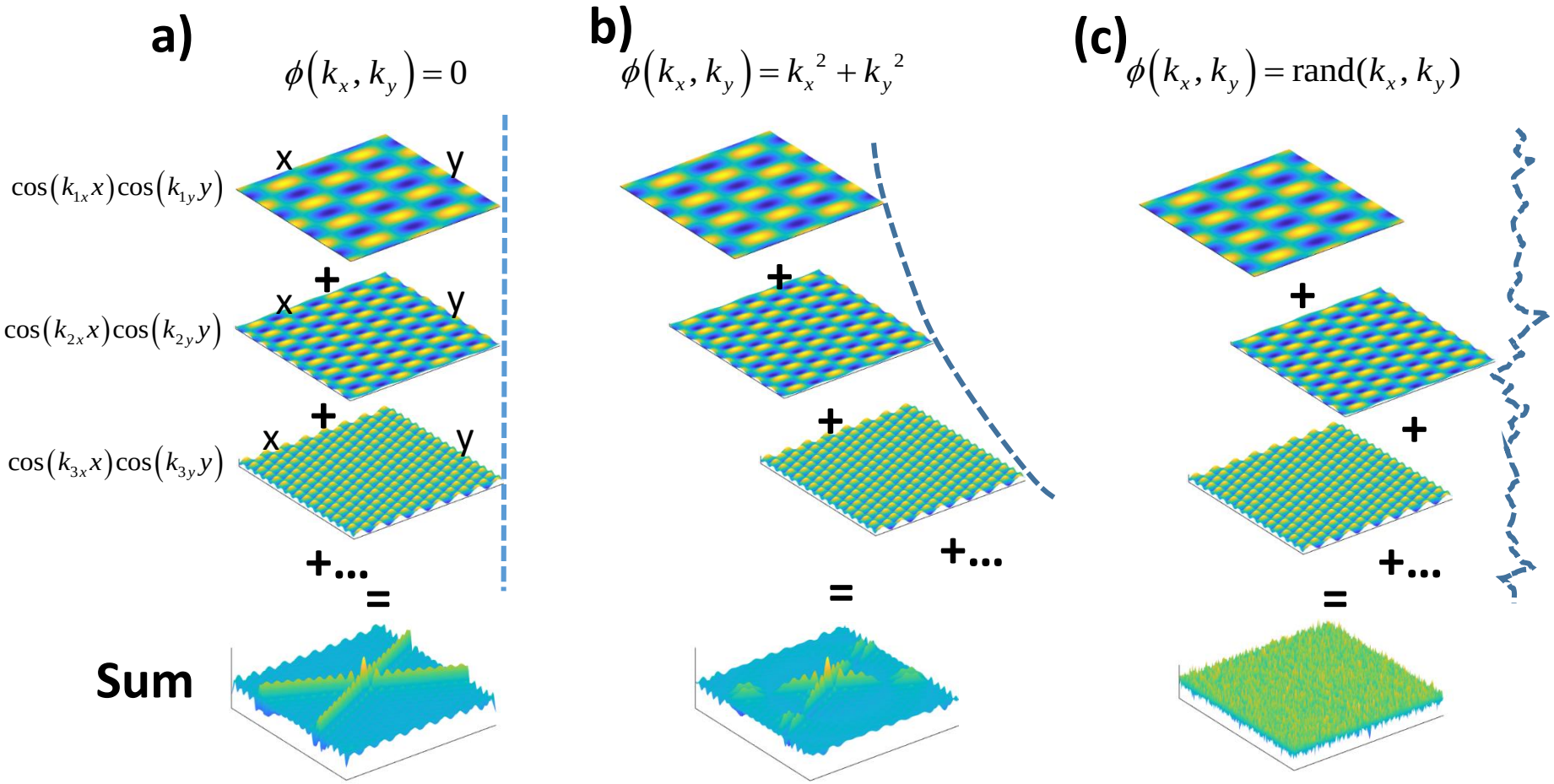


Figure 5.2. *Significance of the spectral phase in 2D. A series of sinusoids (only 3 are shown here for clarity) are added up with various amounts of relative shifts (spectral phases), as follows. a) The individual frequency components are aligned, meaning that there are no relative spatial shifts, as illustrated by a vertical dash line. The result of summing all frequency components is shown in the bottom graph. b) The spectral components are shifted in space according to a quadratic function, as suggested by the parabolic dash line. The resulting signal is broader than in a). c) The frequency components are summed up with random spatial shifts, as indicated by the irregular dash line. The resulting signal is random as well.*

5.2 Significance of the Spectral Phase

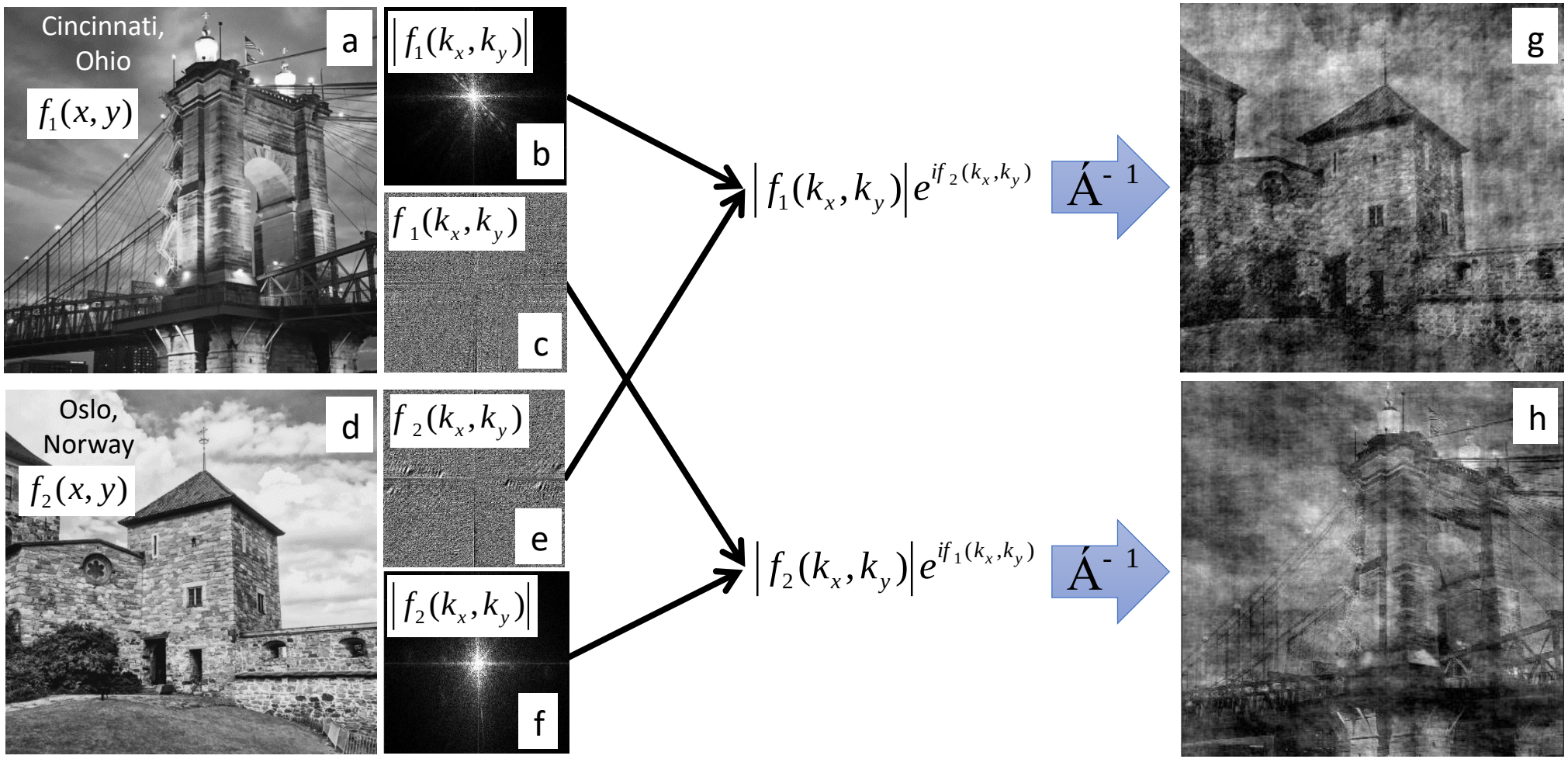


Figure 5.3. *Two different images have their respective spectral phases swapped. a) Image 1, signal f_1; b) magnitude of the Fourier transform of f_1; c) spectral phase of f_1; d-f) same as a-c but for image 2, i.e., signal f_2 ; g) image obtained by inverse Fourier transforming the frequency signal consisting of spectral amplitude 1, $|f_1(k_x,k_y)|$, and spectral phase 2, $\phi_2(k_x,k_y)$; h) image obtained by inverse Fourier transforming $|f_2(k_x,k_y)|e^{i\phi_1(k_x,k_y)}$. The results show that, in this case, the spectral phase, rather than amplitude, dictates the appearance of the images. Courtesy of Chenfei Hu.*

5.3 Properties Specific to 2D Functions

a) Shear theorem

If *f(x,y)* is sheared then its transform is sheared to the same degree in the perpendicular direction.

$$f(x+by, y) \rightarrow \tilde{f}(k_x, k_y - bk_x)$$

<u>Proof</u>

Let us change variables to $u = x + by$

$$v = y$$

The Jacobian for this variable change is $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{dy} \end{vmatrix} = \begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} = 1$

5.3 Properties Specific to 2D Functions

a) Shear theorem

Proof

Thus

$$dudv = J\, dxdy$$

$$= dxdy$$

The Fourier transform of the sheared function is

$$\tilde{f}(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+by, y) e^{-i(k_x \cdot x + k_y \cdot y)} dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v) e^{-i\left[k_x (u-bv) + k_y \cdot v\right]} dudv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v) e^{-i\left[k_x \cdot u + v(k_y - bk_x)\right]} dudv$$

$$= \tilde{f}(k_x, k_y - bk_x) \qquad (q.e.d.)$$

5.3 Properties Specific to 2D Functions

b) Rotation theorem

If *f(x,y)* is rotated in the *(x,y)* plane, then its Fourier transform is rotated in the (k_x,k_y) plane by the same angle, in the same direction

The rotated coordinates are $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

The Rotation theorem states

$$f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \leftrightarrow \tilde{f}(k_x\cos\theta - k_y\sin\theta, k_x\sin\theta + k_y\cos\theta)$$

5.3 Properties Specific to 2D Functions

b) Rotation theorem

<u>Proof</u>

The Fourier Transform of the rotated function is

$$\tilde{f}_\theta(k_x, k_y) = \iint f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) e^{-i(k_x \cdot x + k_y \cdot y)} dxdy$$

Using a change of variables we get,

$$u = x\cos\theta - y\sin\theta$$
$$v = x\sin\theta + y\cos\theta$$
$$x = u\cos\theta + v\sin\theta$$
$$y = -u\sin\theta + v\cos\theta$$

5.3 Properties Specific to 2D Functions

b) Rotation theorem

<u>Proof</u>

Using the Jacobian we have $dudv = \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} dxdy$

$$= dxdy$$

Thus,

$$f_\theta(k_x, k_y) = \iint f(u,v) e^{-i\left[k_x(u\cos\theta + v\sin\theta) + k_y(-u\sin\theta + v\cos\theta)\right]} dudv$$

$$= \iint f(u,v) e^{-i\left[u(k_x\cos\theta - k_y\sin\theta) + v(k_x\sin\theta + k_y\cos\theta)\right]} dudv$$

$$= \tilde{f}(k_x\cos\theta - k_y\sin\theta, k_x\sin\theta + k_y\cos\theta)$$

$$(q.e.d.)$$

5.3 Properties Specific to 2D Functions

c) Affine theorem

Affine transformation changes the vector *(x,y)* into *(ax+by+c,dx+ey+f)*

Here a,b,c,d,e and f are <u>scalars</u> ⇒ transformation is linear followed by a shift

If an image undergoes affine transformation, then

- Points that were collinear remain collinear
- Ratios of distance along a line <u>do not</u> change upon transformation i.e $\frac{|p_2 - p_1|}{|p_3 - p_2|} = const$

  where p1, p2 and p3 are collinear points

Proof- Make use of the shift, similarity and shear theorems to prove

5.4 Extension of 1D Properties

1) *Central Ordinate theorem*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \tilde{f}(0,0)\,.$$

2) *Shift theorem*

$$f(x-a, y-b) \leftrightarrow e^{-i(k_x \cdot a + k_y \cdot b)}\,\tilde{f}(k_x, k_y)\,.$$

3) Parseval's theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| f(x,y) \right|^2 dxdy = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| f(k_x, k_y) \right|^2 dk_x dk_y\,.$$

5.4 Extension of 1D Properties

4) *Similarity theorem*

$$f(ax,by) \rightarrow \frac{1}{|ab|}\tilde{f}\left(\frac{k_x}{a},\frac{k_y}{b}\right).$$

5) *Convolution theorem*

$$f(x,y) \circledV_{xy} g(x,y) \leftrightarrow \tilde{f}(k_x,k_y)\tilde{g}(k_x,k_y).$$

6) Correlation theorem

$$f(x,y) \otimes_{xy} g(x,y) \leftrightarrow \tilde{f}(k_x,k_y)\tilde{g}^*(k_x,k_y).$$

5.4 Extension of 1D Properties

7) *Differentiation theorem*

$$\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n f(x,y) \rightarrow (ik_x)^m (ik_y)^n \tilde{f}(k_x,k_y) \qquad a)$$

$$\left[\left(\frac{\partial}{\partial x}\right)^m + \left(\frac{\partial}{\partial y}\right)^n\right] f(x,y) \leftrightarrow \left[(ik_x)^m + (ik_y)^n\right] \tilde{f}(k_x,k_y) \qquad b).$$

8) *Moment theorem*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy = i\frac{\partial \tilde{f}}{\partial k_x}(0,0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy = i\frac{\partial \tilde{f}}{\partial k_y}(0,0).$$

- *Second and higher order moments can also be obtained similarly using the differentiation theorem*

- 1st order moment (center of gravity)

$$\langle x \rangle = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x, y)dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy} = i\frac{\partial \tilde{f} / \partial k_x(0, 0)}{\tilde{f}(0, 0)}$$

$$\langle y \rangle = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x, y)dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy} = i\frac{\partial \tilde{f} / \partial k_y(0, 0)}{\tilde{f}(0, 0)}.$$

- 2nd order moment

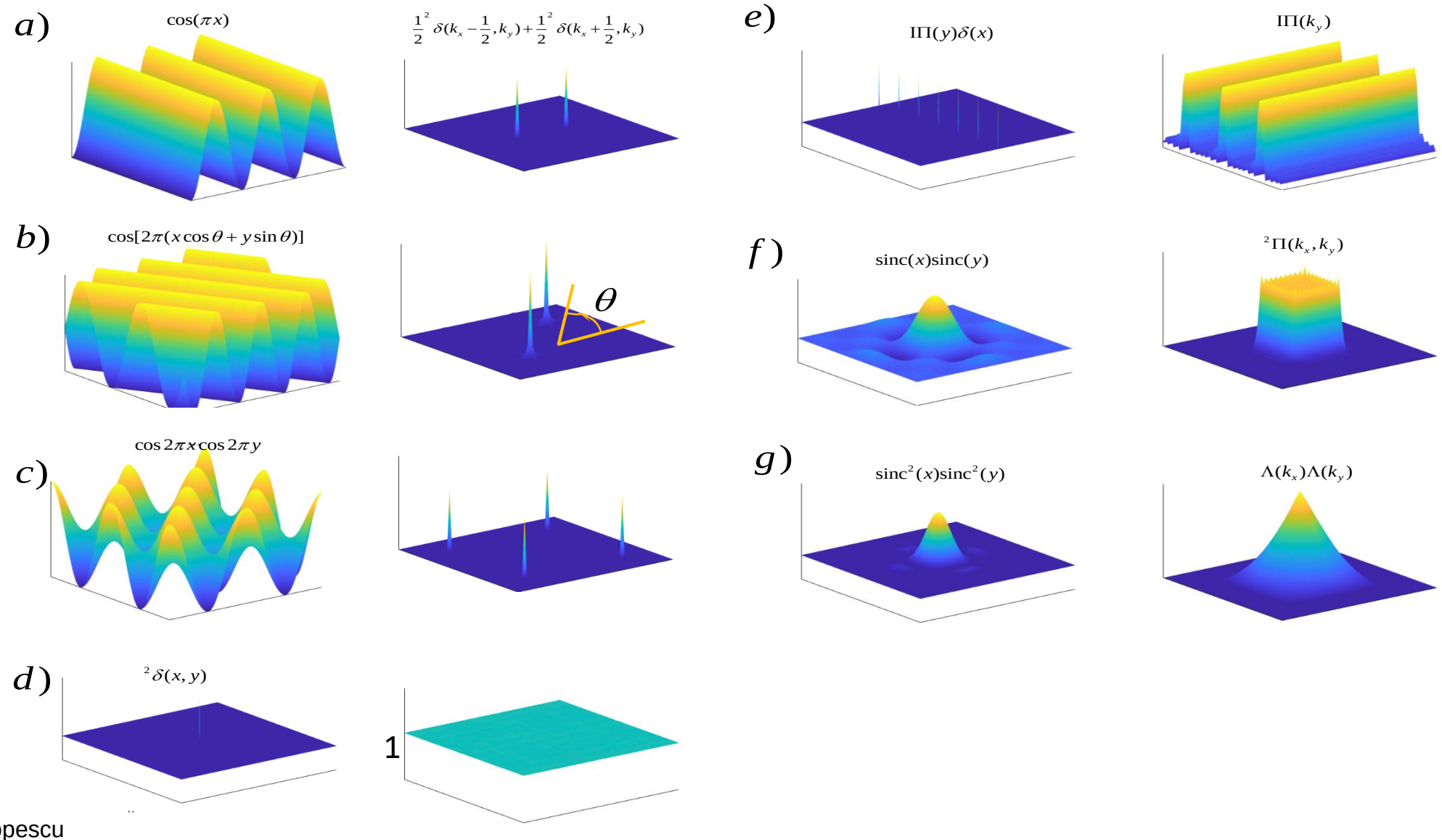
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x, y)dxdy = -\frac{\partial^2 \tilde{f}}{\partial {k_x}^2}(0, 0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x, y)dxdy = -\frac{\partial^2 \tilde{f}}{\partial {k_y}^2}(0, 0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x, y)dxdy = -\frac{\partial^2 \tilde{f}}{\partial k_x \partial k_y}(0, 0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2 + y^2) f(x, y)dxdy = -\left[\frac{\partial^2 \tilde{f}}{\partial {k_x}^2}(0, 0) + \frac{\partial^2 \tilde{f}}{\partial {k_y}^2}(0, 0)\right].$$

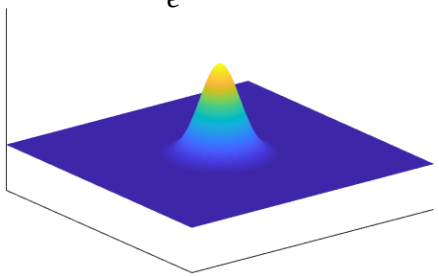
5.5 Common 2D Fourier Transform Pairs



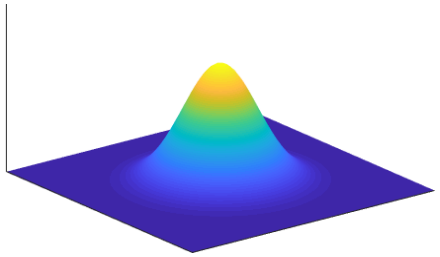
5.5 Common 2D Fourier Transform Pairs

$a)$

$e^{[-\pi(x^2+y^2)]}$

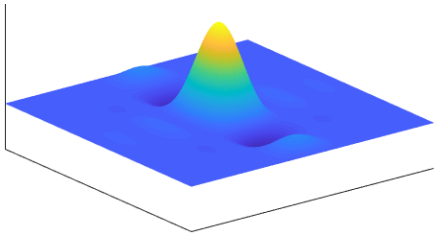


$e^{[-\pi(k_x{}^2+k_y{}^2)]}$

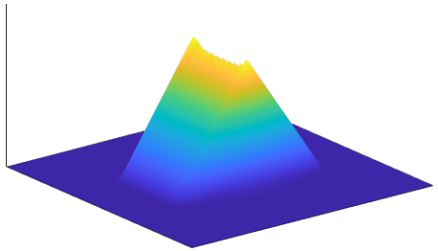


$b)$

$\mathrm{sinc}^2(x)\mathrm{sinc}(y)$

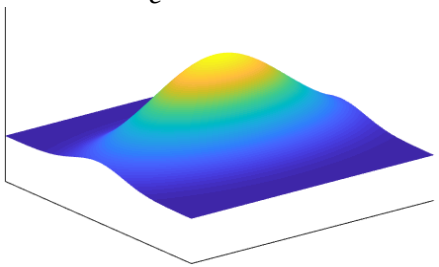


$\Lambda(k_x)\Pi(k_y)$

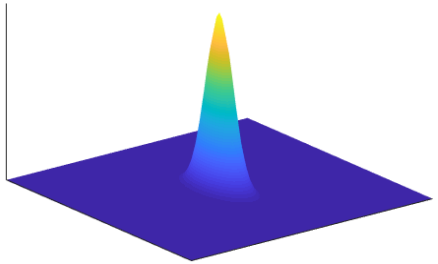


$c)$

$e^{[-\pi(\frac{x^2}{A^2}+\frac{y^2}{a^2})]}$

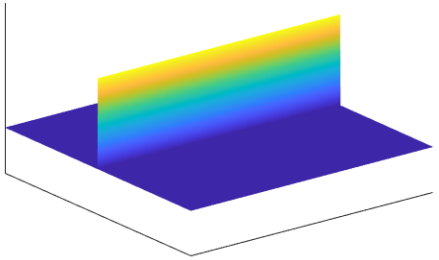


$Aae^{[-\pi(A^2k_x{}^2+a^2k_y{}^2)]}$

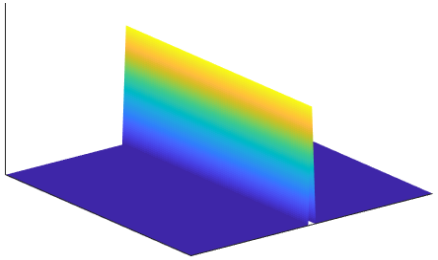


$d)$

$\delta(y)$

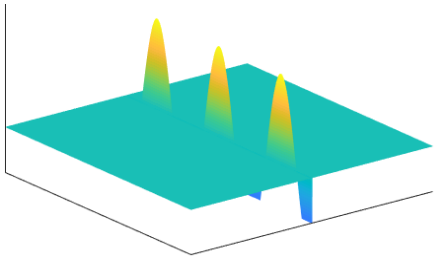


$\delta(k_x)$

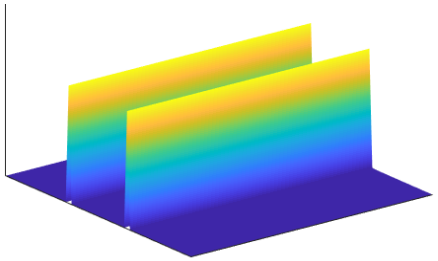


$e)$

$\cos \pi y \delta(x)$



$\Pi(k_y)$



5.6 Polar coordinates: the Hankel Transform

- Cartesian to polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\frac{y}{x}.$$

$$k_x = k\cos\theta'$$
$$k_y = k\sin\theta'$$
$$k = \sqrt{k_x{}^2 + k_y{}^2}$$
$$\theta' = \tan^{-1}\frac{k_y}{k_x}.$$

5.6 Hankel Transform

- Most optical systems exhibit circular symmetry which can be utilized to reduce a 2D Fourier Transform into a 1D integral
  - non-trivial variable is the radial coordinate ***r***

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i\left(k_x\cdot x+k_y\cdot y\right)}dxdy = \int_{0}^{\infty}\int_{0}^{2\pi} f(r)e^{-ikr\cos(\theta-\theta')}rdrd\theta = \int_{0}^{\infty} f(r)\left[\int_{0}^{2\pi} e^{-ikr\cos\theta}d\theta\right]rdr$$

  - Bessel function of zeroth order and first kind:

$$J_0(kr) = \frac{1}{2\pi}\int_{0}^{2\pi} e^{-ikr\cos\theta}d\theta\,.$$

- This results in a Hankel Transform relationship

$$\tilde{f}(k) = 2\pi\int_{0}^{\infty} f(r)J_0(kr)rdr \qquad f(r) = \frac{1}{2\pi}\int_{0}^{\infty}\tilde{f}(k)J_0(kr)kdk\,.$$

5.6 Hankel Transform

- Identities of Bessel function of the first kind
  - *n^{th} order Bessel function*

$$J_n(x) = \frac{1}{2\pi i^n}\int_0^{2\pi} e^{ix\cos\theta}\cdot e^{in\theta}d\theta$$

  - Recurrence relation

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

  - Normalization

$$\int_0^{\infty} J_n(x)dx = 1.$$

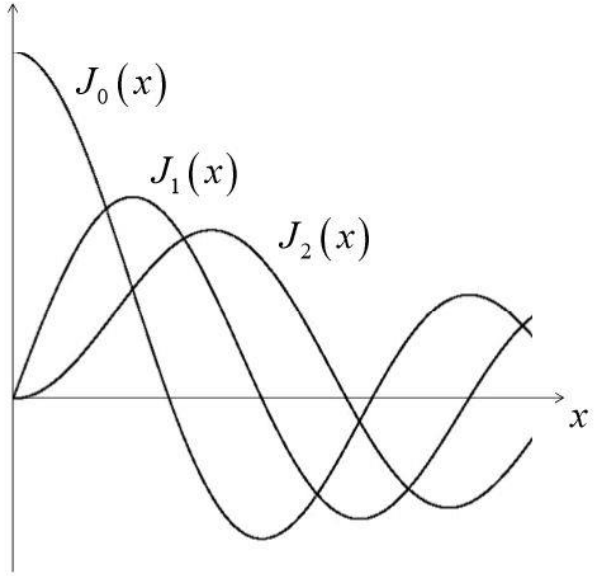


Figure 5.6. *Bessel functions of orders n=0,1,2*

- Bessel functions of arbitrary order *n* form an orthogonal basis

$$\int_0^{\infty} J_n(kr)J_n(k'r)rdr = \frac{\delta(k-k')}{k}.$$

5.6 Hankel Transform theorems

1) Central Ordinate theorem
   - Fourier transform of origin equals the area under the function

$$\tilde{f}(0) = 2\pi \int_0^\infty f(r) J_0(0) r dr = 2\pi \int_0^\infty f(r) r dr$$

2) Shift theorem
   - Hankel transform does not apply since the circular symmetry is destroyed during shifting

3) Parseval's theorem

$$\int_0^\infty |f(r)|^2 r dr = \frac{1}{2\pi} \int_0^\infty |\tilde{f}(k)|^2 k dk .$$

5.6 Hankel Transform theorems

4) Similarity theorem

- Note that the scaling factor indicates that the Fourier Transform is in 2D space

$$f(ar) \rightarrow \frac{1}{|a|^2}\tilde{f}\left(\frac{k}{a}\right).$$

5) Convolution theorem

- Upon expressing the convolution integral polar coordinates we get,

$$\int_0^\infty \int_0^{2\pi} f(r')g(\rho)r'dr'd\theta \leftrightarrow \tilde{f}(k)\tilde{g}(k)$$ where $\left(\rho^2 = r^2 + r'^2 - 2rr'\cos\theta\right)$.

- Since the integral depends on $\theta \rightarrow$ it cannot be expressed as a hankel transform

5.6 Hankel Transform theorems

6) Laplacian

- Fourier representation of the Laplacian transform

$$\nabla^2 f(r) = \frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} \leftrightarrow -k^2 \tilde{f}(k).$$

7) Second moment

$$\int_0^\infty r^2 f(r) r dr = -\frac{\tilde{f}''(0)}{2\pi}.$$

5.7 Common Hankel Transform Pairs

$f(r)$	$\tilde{f}(k)$
$\delta(x,y)=\dfrac{\delta(r)}{\pi r}$	1
$f\left(\dfrac{r}{a}\right)$	$\lvert a\rvert^2\,\tilde{f}(ak)$
$\delta(r-a),\ a>0$	$2\pi a J_0(ka)$
$\Pi\left(\dfrac{r}{2a}\right)$	$2\pi\lvert a\rvert^2\,\dfrac{J_1(ka)}{k}$
$\dfrac{1}{r}$	$\dfrac{1}{k}$
$e^{-\alpha r},\ \alpha=a+ib$	$\dfrac{2\pi\alpha}{\left(k^2+\alpha^2\right)^{3/2}}$
$\dfrac{e^{-\alpha r}}{r}$	$\dfrac{2\pi\alpha}{\left(k^2+\alpha^2\right)^{1/2}}$
$e^{-\frac{\alpha r^2}{2}}$	$e^{-\frac{k^2}{2\alpha}}$

5.8 Fourier-slice theorem

- Fourier properties can be used to reconstruct knowledge of its projections along certain directions
  - Example: X-ray CT
- Fourier transform of the projection along the y-axis is equal to a slice of from the 2D Fourier transform of the object function
  - This is the **Fourier-Slice Theorem**
  - This theorem can also be generalized to 3D objects

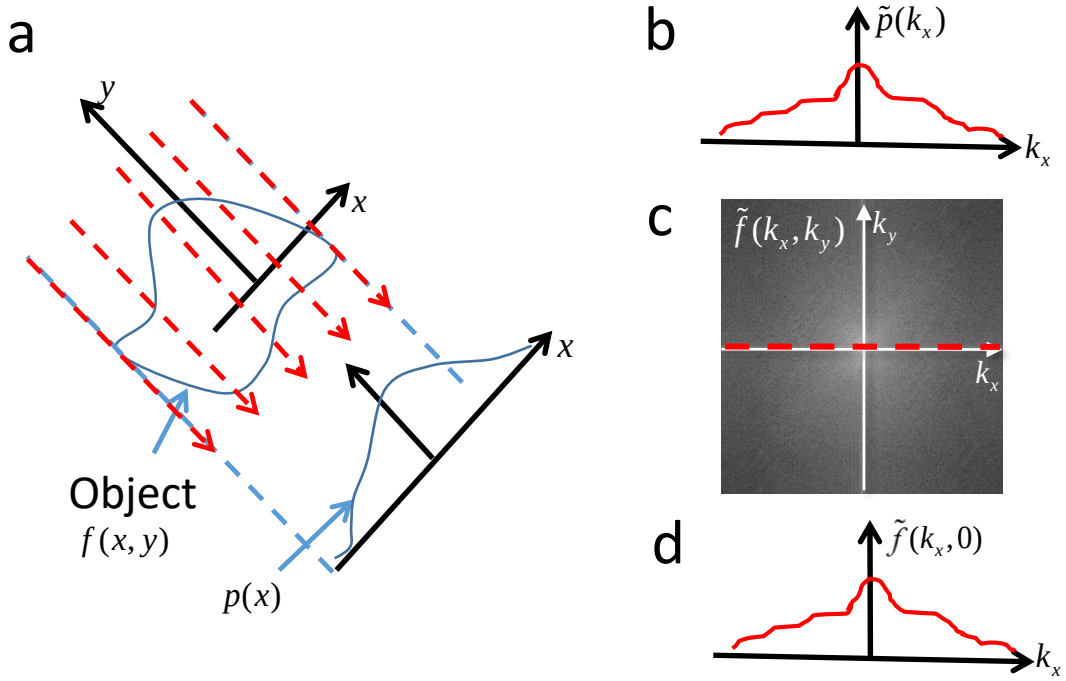


Figure 5.8 *a) Projecting the 2D object along the y-axis yields a 1D signal, p(x). b) The Fourier transform of p(x). c) The 2D Fourier transform of the object. d) The profile along the dash line in c), i.e., $f_T(k_x,0)$. Note that the profiles in b) and d) are identical, meaning that the Fourier transform of the projection gives a "slice" of the 2D Fourier transform.*

5.8 Fourier-slice theorem

- If $p(x) = \int f(x, y)dy$ is the projection along y-axis and *f(x, y)* is the object function then

$$\tilde{p}(k_x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dye^{ik_x x}dx =$$

$$= \int_{-\infty}^{\infty} f(k_x, y)dy = f(k_x, k_y)\Big|_{k_y=0} .$$

- Multiple projections along different directions (θ) can be acquired to obtain the entire Fourier transform
  - Operation of integrating along different directions to obtain the projections is called the **Radon Transform,** i.e. *p(x, θ)* is the radon transform of *f(x, y)*
- Transparent objects can also be studied using this technique where phase delay is observed instead of absorption

Chapter 6 – 3D Fourier Transform

6.1 3D Fourier Transform

- Fourier Transforms in 3D are given as

$$f(x,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}\left(k_x,k_y,k_z\right)e^{i(k_x\cdot x+k_y\cdot y+k_z\cdot z)}dk_x dk_y dk_z$$

$$\tilde{f}(k_x,k_y,k_z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y,z)e^{-i(k_x\cdot x+k_y\cdot y+k_z\cdot z)}dxdydz\,.$$

- Wavefronts are given by

$\phi(x,y,z)=k_x x+k_y y+k_z z$ where $\mathbf{k}=(k_x,k_y,k_z)$

  - Note that the direction of wavefront makes an angle perpendicular to **k,** that is consistent with

$$\nabla\phi(x,y,z)=\mathbf{k}$$

6.1 3D Fourier Transform

- 3D Analogy of convolution

$$f \circledv_{xyz} g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y',z')g(x-x',y-y',z-z')dx'dy'dz'$$

- Similarly cross correlation is

$$f \otimes_{xyz} g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y',z')g(x'-x,y'-y,z-z')dx'dy'dz'$$

  - Similar to 1D and 2D cases, the difference between correlation and convolution remains whether *g* is rotated about origin or not

- Also when *f* has 3 separate variables $\rightarrow$ $f(x,y,z) = f_1(x)f_2(y)f_3(z)$ then

$$\tilde{f}(k_x,k_y,k_z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(k_x x+k_y y+k_z z)}dxdydz = \int_{-\infty}^{\infty} f_1(x)e^{-i(k_x x)}dx \int_{-\infty}^{\infty} f_2(y)\cdot e^{-i(k_y y)}dy \int_{-\infty}^{\infty} f_3(z)\cdot e^{-i(k_z z)}dy$$

6.2 Extension of 1D Properties

1) *Central Ordinate theorem*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(x,y,z\right)dxdydz = \tilde{f}(0,0,0)\,.$$

2) *Shift theorem*

$$f(x-a,y-b,z-c) \leftrightarrow e^{-i(k_x\cdot a+k_y\cdot b+k_z\cdot c)}\,\tilde{f}(k_x,k_y,k_z)\,.$$

3) Parseval's theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x,y,z)\right|^2 dxdydz = \frac{1}{\left(2\pi\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\tilde{f}(k_x,k_y,k_z)\right|^2 dk_x dk_y dk_z\,.$$

6.2 Extension of 1D Properties

4) *Similarity theorem*

$$f(ax,by,cz) \leftrightarrow \frac{1}{|abc|}\tilde{f}\left(\frac{k_x}{a},\frac{k_y}{b},\frac{k_z}{c}\right).$$

5) *Convolution theorem*

$$f(x,y,z)\textcircled{V}_{xyz}g(x,y,z) \leftrightarrow \tilde{f}(k_x,k_y,k_z)\tilde{g}(k_x,k_y,k_z).$$

6) Correlation theorem

$$f(x,y,z)\otimes_{xyz} g(x,y,z) \leftrightarrow \tilde{f}(k_x,k_y,k_z)\tilde{g}^*(k_x,k_y,k_z).$$

6.2 Extension of 1D Properties

7) *Differentiation theorem*

$$\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n \left(\frac{\partial}{\partial z}\right)^p f(x,y,z) \leftrightarrow (ik_x)^m (ik_y)^n (ik_z)^p \tilde{f}(k_x,k_y,k_z).$$

$$\left[\left(\frac{\partial}{\partial x}\right)^m + \left(\frac{\partial}{\partial y}\right)^n + \left(\frac{\partial}{\partial z}\right)^p\right] f(x,y,z) \leftrightarrow \left[(ik_x)^m + (ik_y)^n + (ik_z)^p\right] \tilde{f}(k_x,k_y,k_z).$$

8) *Moment theorem*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y,z)dxdydz = i\frac{\partial \tilde{f}}{\partial k_x}(0,0,0) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} zf(x,y,z)dxdydz = i\frac{\partial \tilde{f}}{\partial k_z}(0,0,0).$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y,z)dxdydz = i\frac{\partial \tilde{f}}{\partial k_y}(0,0,0)$$

- *Second and higher order moments can also be obtained similarly using the differentiation theorem*

6.3 Spectral Phase

- Similar to 1D and 2D Fourier Transforms, spectral phase contains significant information about the original signal
- Spectral phase determines how the different frequency components are arranged with respect to each other
- Therefore, summing up the same frequency components with different relative phases gives a different signal
  - For example, if 2 input functions are Fourier transformed and have the spectral phases switched before taking the inverse Fourier transform gives very different outputs with respect to the initial functions

6.4 Cylindrical Coordinates

- Cylindrical coordinates are useful in situations where the problems contain symmetry, such as circular and cylindrical symmetries
- Consider an arbitrary function *f* then,

$$f(x,y,z) = g(r_\perp, \theta, z)$$

$$\tilde{f}(k_x, k_y, k_z) = \tilde{g}(k_\perp, \theta', k_z)$$

where,

$$x + iy = r_\perp e^{i\theta}$$

$$k_x + ik_y = k_\perp e^{i\theta'}$$

- *Therefore Fourier transform relation for g,*

$$g(r_\perp, \theta, z) = \int_{-\infty}^{\infty}\int_{0}^{2\pi}\int_{0}^{\infty} \tilde{g}(k_\perp, \theta', k_z) e^{i\left[k_\perp r_\perp \cos(\theta-\theta') + k_z \cdot z\right]} dk_\perp d\theta' dk_z$$

$$\tilde{g}(k_\perp, \theta', k_z) = \int_{-\infty}^{\infty}\int_{0}^{2\pi}\int_{0}^{\infty} g(r_\perp, \theta, z) e^{-i\left[k_\perp r_\perp \cos(\theta-\theta') + k_z \cdot z\right]} dr_\perp d\theta dz\,.$$

6.4 Cylindrical Coordinates – Circular Symmetry

- Circular symmetry is when *f* is independent of *θ* and thus $\tilde{f}$ is independent of *θ'*
  - *This symmetry implies that functions depend only on magnitude of the transverse spatial coordinate and frequency*

$$f\left(x,y,z\right)=h\left(r_{\perp},z\right)$$

$$\tilde{f}\left(k_x,k_y,k_z\right)=\tilde{h}\left(k_{\perp},k_z\right)$$

- Circular symmetry allows us to utilize the results of the Hankel transform,

$$h(r_{\perp},z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\tilde{h}\left(k_{\perp},k_z\right)J_0\left(k_{\perp}r_{\perp}\right)e^{ik_z\cdot z}k_{\perp}dk_{\perp}dk_z$$

$$\tilde{h}(k_{\perp},k_z)=2\pi\int_{-\infty}^{\infty}\int_{0}^{\infty}h\left(r_{\perp},z\right)J_0\left(k_{\perp}r_{\perp}\right)e^{-ik_z\cdot z}r_{\perp}dr_{\perp}dz\,.$$

6.4 Cylindrical Coordinates – Cylindrical Symmetry

- Cylindrical symmetry is when *f* is independent of *θ* and *z* and thus $\tilde{f}$ is independent of *θ'*
  - *This symmetry implies that functions can be simplified as*

$$f(x,y,z) = p(r_\perp)$$

$$\tilde{p}(k_\perp, k_z) = \tilde{p}_1(k_\perp)\delta(k_z)$$

  - *Since p is constant in z, there is a dependence in k_z through a delta function in the Fourier transform*

- Therefore, the transverse Fourier transform simplifies into

$$p(r_\perp) = \frac{1}{2\pi}\int_0^\infty \tilde{p}_1(k_\perp) J_0(k_\perp r_\perp) k_\perp dk_\perp$$

$$\tilde{p}_1(k_\perp) = 2\pi\int_0^\infty p(r_\perp) J_0(k_\perp r_\perp) r_\perp dr_\perp \quad .$$

6.5 Spherical Coordinates

- Similar to cylindrical coordinates, spherical coordinates also turn out to be useful in taking advantage of symmetries
  - *For an arbitrary function f, the transformation from cartesian to spherical coordinates*

$$f(x,y,z) = g(r,\theta,\phi) \qquad x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

$$\tilde{f}(k_x,k_y,k_z) = \tilde{g}(k,\theta',\phi'). \qquad k_x = k\sin\theta'\cos\phi',\ k_y = k\sin\theta'\sin\phi',\ k_z = k\cos\theta'$$

- Resulting Fourier integrals are

$$g(r,\theta,\phi) = \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} \tilde{g}(k,\theta',\phi')\, e^{ikr\left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right]} k^2 \sin\theta'\, dk\, d\theta'\, d\phi'$$

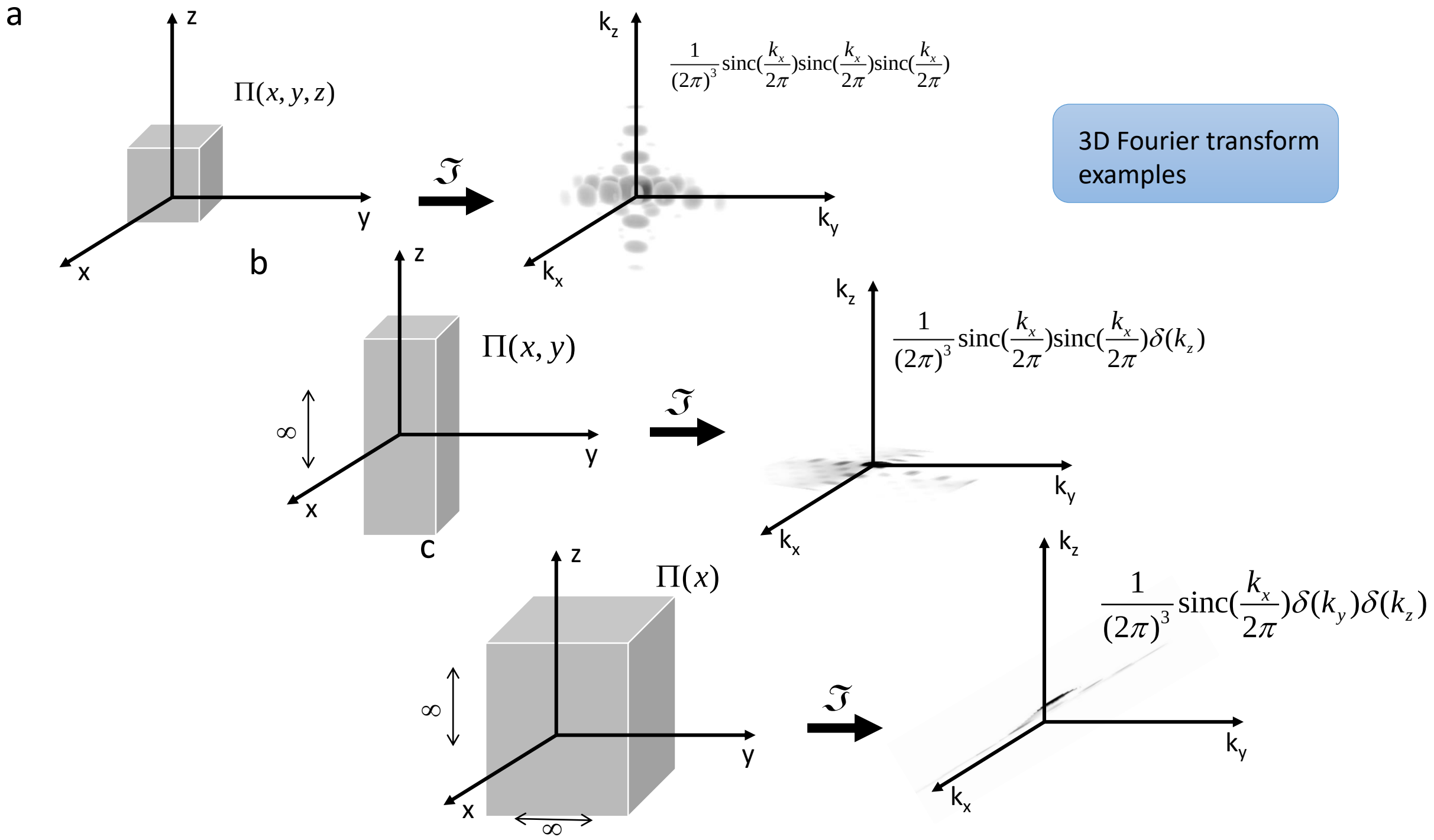
$$\tilde{g}(k,\theta',\phi') = \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} g(r,\theta,\phi)\, e^{-ikr\left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right]} r^2 \sin\theta\, dr\, d\theta\, d\phi\,.$$

Common Fourier Transforms

$\delta(x-a, y-b, z-c)$	$e^{-i(k_x \cdot x + k_y \cdot y + k_z \cdot z)}$
$\Pi(x, y, z)$ **(cube)**	$\frac{1}{(2\pi)^3}\operatorname{sinc}\left(\frac{k_x}{2\pi}\right)\operatorname{sinc}\left(\frac{k_y}{2\pi}\right)\operatorname{sinc}\left(\frac{k_z}{2\pi}\right)$
$\Pi(x, y)$ **(bar)**	$\frac{1}{(2\pi)^2}\operatorname{sinc}\left(\frac{k_x}{2\pi}\right)\operatorname{sinc}\left(\frac{k_y}{2\pi}\right)\delta(k_z)$
$\Pi(x)$ **(slab)**	$\frac{1}{2\pi}\operatorname{sinc}\left(\frac{k_x}{2\pi}\right)\delta(k_y)\delta(k_z)$
$\Pi\left(\frac{r}{2}\right)$ **(ball)**	$\frac{\sin k - k \cdot \cos k}{2k^3}$

Common Fourier Transforms

$\Pi(x)\Pi\left(\sqrt{y^2+z^2}\right)$ **(disk)**	$\frac{1}{(2\pi)^3}\operatorname{sinc}\left(\frac{k_x}{2\pi}\right)\frac{J_1\left(\sqrt{k_y^{\,2}+k_z^{\,2}}\big/2\right)}{\sqrt{k_y^{\,2}+k_z^{\,2}}\big/\pi}$
$\frac{e^{i\alpha r}}{4\pi r},\ \alpha>0$	$\frac{1}{\alpha^2-k^2}$
$\frac{e^{-r/R}}{\frac{4}{3}\pi R^3}$	$\frac{1}{(2\pi)^3}\frac{6}{\left(1+q^2R^2\right)^2}$
$e^{-\frac{\alpha r^2}{2}}$	$e^{-\frac{k^2}{2\alpha}}$


a
z
y
x
$\Pi(x,y,z)$
$\mathfrak{F}$
k_z
k_y
k_x
$\frac{1}{(2\pi)^3}\mathrm{sinc}(\frac{k_x}{2\pi})\mathrm{sinc}(\frac{k_x}{2\pi})\mathrm{sinc}(\frac{k_x}{2\pi})$
3D Fourier transform examples
b
z
y
x
$\Pi(x,y)$
∞
$\mathfrak{F}$
k_z
k_y
k_x
$\frac{1}{(2\pi)^3}\mathrm{sinc}(\frac{k_x}{2\pi})\mathrm{sinc}(\frac{k_x}{2\pi})\delta(k_z)$
c
z
y
x
$\Pi(x)$
∞
∞
$\mathfrak{F}$
k_z
k_y
k_x
$\frac{1}{(2\pi)^3}\mathrm{sinc}(\frac{k_x}{2\pi})\delta(k_y)\delta(k_z)$

6.6 Fourier Slice Theory

- **Fourier-slice theorem:** There exists a relationship between the projection of f along an axis, and the Fourier transform of f

$$p(x,y) = \int f(x,y,z)dz \quad \xrightarrow{\mathfrak{F}} \quad p(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(x,y,z)dz e^{i\left(k_x x + k_y x\right)} dxdy =$$

$$= \int_{-\infty}^{\infty} \tilde{f}(k_x,k_y y)dy$$

$$= \tilde{f}(k_x,k_y,k_z)\Big|_{k_z=0}$$

6.6 Fourier Slice Theory

- Fourier transform of projections are “slices” of the full 3D Fourier transform
  - 3D object imaged using 2D measurements; in X-ray CT, we measured absorption along one direction (projection)
- Can be used to measure transparent objects
  - Phase delay measured instead of absorption➔ diffraction tomography