Chapter 5 – Interferometry

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5.1 Superposition of Fields

- Interference is the phenomenon by which electromagnetic fields interact with one another.
- Interferometry has various applications from motion sensors to spectroscopy and material characterization.
- Interference is the result of the superposition principle

\[ \mathbf{E}[^{\mathbf{r}}, t] = \sum_{j} \mathbf{E}_{j}[^{\mathbf{r}}, t] \]  \hspace{1cm} (5.1)

- Intensity is the measurable quantity

\[ \mathbf{I}[^{\mathbf{r}}, t] = \left\langle |\mathbf{E}[^{\mathbf{r}}, t]|^2 \right\rangle \]  \hspace{1cm} (5.2)

- Where \( \left\langle \right\rangle \) stands for time (or ensemble) average
5.1 Superposition of Fields

- Consider 2 fields $\overline{E_1}$ and $\overline{E_2}$

$$\overline{I}[\vec{r}, t] = \langle |E_1|^2 \rangle + \langle |E_2|^2 \rangle + \langle |E_1 \cdot E_2^*| \rangle + \langle |E_1^* \cdot E_2| \rangle$$  \hspace{1cm} (5.3)

- Note:
  - the dot product $E_1 \cdot E_2^*$
  - Polarization is critical
  - Parallel polarization offers the most interference
  - $|E_1 \cdot E_2| = |E_1||E_2| \cos \alpha$
  - Assume parallel polarization

$$\Rightarrow I = I_1 + I_2 + 2|E_1||E_2| \cos[\Delta \phi[\vec{r}, \tau]]$$  \hspace{1cm} (5.4)

- $\Delta \phi$ is generally a random variable
- For uncorrelated (incoherent) fields $\langle \cos[\Delta \phi] \rangle \rightarrow 0$ => simplest case is a monochromatic field because it is fully coherent.
5.2 Monochromatic Fields

\[ I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cdot \cos[\omega_0 \tau + \Delta\phi] \]  

(5.5)

\[ \tau = \text{Time difference} \]
\[ \phi = \text{The phase shift} \]
\[ \omega_0 = \text{Frequency} \]

- Fringe contrast (visibility): \[ \gamma = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} \]

(5.6)

\[ \Rightarrow \gamma = \frac{I_1 + I_2 + 2\sqrt{I_1 I_2} \cdot 1 - I_1 - I_2 - 2\sqrt{I_1 I_2} \cdot (-1)}{2(I_1 + I_2)} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} \in (0 : 1) \]

- \[ I_1 = I_2 \Rightarrow \gamma = 1 \]
- \[ I_2 \ll I_1 \Rightarrow \gamma \to 0 \]
5.2 Monochromatic Fields

- For \( I_1 = I_2 = I \):
  \[
  I_t = 2I(1 + \cos[\omega_0 \tau])
  \]
  \[(5.7)\]

- Practical Advantages of Interference:
  
  a) Interference term is \( 2\sqrt{I_1I_2} \cos[\Delta \phi] \) so if \( I_1 \) is too small to be measured directly then \( I_2 \) can act as an amplifier. \( I_1I_2 \) gives a higher sensitivity
5.2 Monochromatic Fields

- **Practical Advantages of Interference:**
  
  b) Interference $\sim \sqrt{I_1 I_2}$ $\Rightarrow$ if $I_1$ is divided by 100, interference is divided by 10 $\Rightarrow$ high dynamic range

  
  c) Imagine we frequency shift $E_1$ by $\Delta \omega = \omega_1 - \omega_2$ $\Rightarrow$

  $\sqrt{I_1 I_2} \cdot \cos[\omega_1 t - \omega_2 t] \sim \cos[\Delta \omega t]$

  $\Rightarrow$ can tune $\Delta \omega$ to high frequency ($> 1$ kHz) $\Rightarrow$ Low Noise
5.3 Wavefront-Division Interferometry

- Interference is obtained between different portions of the same wavefront (next: amplitude-division)

- Young Interferometer
  - The oldest interferometer
  - Small Slits $\rightarrow$ 1° $\delta$-functions: $\delta[y - \frac{d}{2}]$; $\delta[y + \frac{d}{2}]$;

\[ \phi = k \cdot d \rightarrow d \sim \lim \]

\[ \phi = 2\pi \rightarrow \lambda = 700 \text{ microns} \]

\[ \omega_0 \tau + d \]

\[ \frac{\lambda}{1000} \]
5.3 Wavefront-Division Interferometry

- The field at the plane x=0

\[ E = E_1 + E_2 = E_0 \left( \delta \left[ y - \frac{d}{2} \right] + \delta \left[ y + \frac{d}{2} \right] \right) \]  

(5.8)

- Assume the observation plane is in the far zone => Fraunhoffer diffraction (Fourier)

\[ E[q_y] = \mathcal{F}[E[y]] = E_0 \left[ e^{ig_y \frac{d}{2}} + e^{-ig_y \frac{d}{2}} \right] = E_0 \cdot 2 \cos \left[ q_y \frac{d}{2} \right] \]  

(5.9)

- Remember \( q_y = \frac{y'}{\lambda z} \) (chapter 3)

- Fringes \( \rightarrow \cos \left[ \frac{d}{2\lambda z} \cdot y' \right] \)  

(5.10)
5.3 Wavefront-Division Interferometry

- Similarity relationship again:

\[ d_1 \quad I \quad d_2 \]

\[ \text{and } d_2 \quad I \quad d_2 \]

- **Note**: using Fourier it is easy to generalize to arbitrary slit/particle shape; similar to scattering from ensemble of particles => use convolution to express the arbitrary shape.
5.3 Wavefront-Division Interferometry

- We can use convolution to express arbitrary shape:

\[ a \begin{array}{c} \rightarrow \\ d/2 \end{array} \begin{array}{c} \rightarrow \\ d/2 \end{array} \begin{array}{c} \rightarrow \\ y \end{array} = \begin{array}{c} \rightarrow \\ d/2 \end{array} \begin{array}{c} \rightarrow \\ d/2 \end{array} \begin{array}{c} \rightarrow \\ y \end{array} \]

- Other Wavefront-Division Interferometers

  a) Fresnel Mirrors

  Image S thru \( M_1 \) and \( M_2 \)
  \( \rightarrow \) virtual sources \( S_1 \) and \( S_2 \)
  \( \rightarrow \) \( S_1 \) and \( S_2 \) act as Young’s pinholes
  \( \rightarrow \) same equations
  \( \rightarrow \) \( S_1 \) and \( S_2 \) are derived from the same sources and therefore coherent
5.3 Wavefront-Division Interferometry

b) **Lloyd’s Mirror**

- $S$ and $S' \rightarrow$ Young
5.3 Wavefront-Division Interferometry

c) Thin films:

- Films of oil break the white light into colors due to interference and phase $= f(\lambda)$

\[ S \quad \quad P \quad \quad n \]

\[ h \quad \quad \quad \quad \quad \quad k \]

- Localized on the surface of lens

\[ \text{Circular fringes} = \text{rings} \]
5.4 Amplitude-Division Interferometry

- Interference is obtained by replicating the wavefront => less amplitude in each beam.
- Michelson Interferometer:

\[ \phi(t) = k \Delta L \]

- Very sensitive to path length differences between ‘arms’
- Eg. It has been used to measure pressure in rarefied gases (place cell on one arm -> produce \( \Delta \phi \))

\[ \omega = k \cdot v = \phi \]
5.4 Amplitude-Division Interferometry

- BS- beam splitter
  - assume thin for now!
- Let $L_1, L_2$ be the lengths of the 2 areas
  => The intensity at the detector:

$$I(\Delta L) = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \left(2\pi \frac{\Delta L}{\lambda}\right)$$  \hspace{1cm} (5.11)

Note: $\Delta L = L_2 - L_1 = C(t_2 - t_1) = C\tau$

$\tau = \text{time delay} \Rightarrow \cos(\omega \tau)$

- We’ll come back to Michelson with low-coherence light, temporal coherence, OCT, etc
5.4 Amplitude-Division Interferometry

- Other amplitude-division interferometry
  a) Plane parallel plate $\rightarrow$ reflection.

  - point source-
  - inter-fringe $= f(n, d) \rightarrow$ metrology

b) Plane parallel plate- transmission.

  - Note: With plane wave incident, fringes are localized at infinity $\Rightarrow$ Need lens
5.4 Amplitude-Division Interferometry

**c) Fizeau Interferometer:**

\[ Widrelson \sim \cos(\omega t) \]
\[ \omega = k \cdot v \]
\[ H - Z - \cos(k\Delta x) \]

- Used to test mirrors and other surfaces.

**d) Mach-Zender Interferometer**

- Very Common
- Shear interferometry
  - Tilt one mirror
  - Fringes → analysis of surfaces
5.5 Multiple Beam Interference

Fabry-Penot interferometry

\[
\frac{I^{(r)}}{I^{(i)}} = \frac{F \sin^2 \frac{\delta}{2}}{1 + F \sin^2 \frac{\delta}{2}};
\]

\[
I^{(t)} = I^{(i)} - I^{(r)}
\]

\[
F = \frac{4R}{(1 - R)^2} \Rightarrow \text{Finess coeff. (5.12)}
\]

\[
\delta = 2k_0 d \cdot n \cdot \cos \theta \Rightarrow \text{one pass phase shift}
\]
5.5 Multiple Beam Interference

- Transmitted Intensity

\[
\frac{I^{(r)}}{I^{(i)}} \quad \text{with} \quad R_1, R_2 \rightarrow \text{reflectivities}
\]

- R increases → narrower lines

- i.e. More reflection orders participate in interference

- In practice, Fabry-Perot gave accurate information about spectral lines (also called etalon)

\[
2k\pi \rightarrow 2k\pi + 2\pi
\]

- Rayleigh Criterion:

\[
\Delta \text{ is fixed} \Rightarrow \text{etalon}
\]

\[
2 \text{ lines are separated if:} \quad \delta \lambda \geq FWHM
\]
5.6 Interference with Partially Coherent Light

- So far, we assumed monochromatic light = fully coherent.
- What happens when an arbitrary, broad-band, extended source is used for interference?
  - a) Michelson interferometry

![Diagram of Michelson interferometer with a graph showing fringe pattern]

- The contrast of the fringe is decreasing
- i.e. limited temporal coherence.
5.6 Interference with Partially Coherent Light

b) Young Interferometer:

\[ I_1 + I_2 \]

\[ \Omega \]

\[ A_c \approx \frac{\lambda^2}{\Omega} \]

- i.e. Limited Spatial Coherence
5.7 Temporal Coherence

- Coherence defines the degree of correlation between fields:
  - Typical correlation at one point in space (typical coherence)
  - Temporal correlation between fields at two points (spatial coherence)
- Given the field at one point $E(r,t)$, the mutual coherence function is:

$$\Gamma(\tau) = \langle E(t) \cdot E^*(t + \tau) \rangle_t$$

$$= \int_{-\infty}^{\infty} E(t) \cdot E^*(t + \tau) \, dt = \text{autocorrelation function}$$

(5.13)
5.7 Temporal Coherence

- Note: \[ \Gamma(0) = \langle |E(t)|^2 \rangle = I \Rightarrow \text{irradiance} \] (5.14)

- So \[ \Gamma = E \otimes E \]

- Apply again the correlation theorem (Eq 1.30)

\[ \Im[\Gamma] = \tilde{E}(\omega)\tilde{E}^*(\omega) = \left| \tilde{E}(\omega) \right|^2 \] (5.15)

\[ = S(\omega) = \text{Spectrum} \Rightarrow \text{FTIR} \]

- So, the autocorrelation function \( \Gamma(\tau) \) relates to the optical spectrum of the field:

\[ \Gamma(\tau) = \int_{-\infty}^{\infty} S(\omega) \cdot e^{-iw\tau} d\omega \quad \text{Wiener-Kintchin theorem.} \] (5.16)
5.7 Temporal Coherence

- Compare 6.16 with 5.28
- It applies even when $E(t)$ does not have a Fourier Transform
- Typically, spectrum is centered on $\omega_0$
- Assume spectrum is $S(\omega - \omega_0)$; apply shift theorem (Eq 1.31)

\[ \Rightarrow \Gamma(\tau) = |\Gamma(\tau)| \cdot e^{i\omega_0 \tau}, \quad (5.17a) \]

where \[ |\Gamma(\tau)| = \int_{-\omega_0}^{\infty} S(u) \cdot e^{iut} du; \quad u = \omega - \omega_0 \quad (5.17b) \]

- Complex degree of coherence

\[ \gamma(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)} \in \mathbb{C} \quad (5.18) \quad \Rightarrow |\gamma(\tau)| \in (0; 1) \quad (5.19) \]
5.7 Temporal Coherence

- Measuring the temporal coherence: Michelson interferometer
  - \( E_{total} = E_1 + E_2 \)
  - Irradiance:
    \[
    I = \langle \left| E_1(t) + E_2(t + \tau) \right|^2 \rangle \quad (5.20)
    \]
  - Assume: \( |E_1| = |E_2| \)
    \[
    I = I_1 + I_{1,2} + \langle E_1(t)E_2^*(t + \tau) \rangle + \langle E_1^*(t)E_2(t + \tau) \rangle
    = 2I_1 + 2\text{Re}[\Gamma(\tau)] \quad (5.21)
    \]
- \( I_{1,2} \) can be measured separately \( \rightarrow \) access to \( \text{Re}(\Gamma) \) directly, by moving \( M_2 \)

\[
\text{Re}[\Gamma(\tau)]
\]

\[
\omega_0
\]
5.7 Temporal Coherence

- The degree of coherence

\[ |\gamma(\tau)| \]

\[ \tau_c = \text{coherence time} \]

\[ \equiv \text{width of } |\gamma(\tau)| \]

\[ \Gamma(\tau) = \Im[S(\omega)] \]

\[ \Rightarrow \Delta\omega \cdot \tau_c = \text{constant} \quad (6.22) \]

\[ \lambda_0 = c \cdot T = c \cdot \frac{2\pi}{\omega_0} ; \Delta\lambda = \Delta(\frac{2\pi c}{\omega}) = 2\pi c \frac{\Delta \omega}{\omega^2} = \lambda \frac{\Delta \omega}{\omega} \Rightarrow \frac{\Delta \lambda}{\lambda} = \frac{\Delta \omega}{\omega} \]

- Interference occurs only if \( \Delta L = l_c \)

- Broad spectrum (\( \Delta \lambda = 100 \text{nm} \)) \( \Rightarrow \) short \( l_c \approx 2 - 3 \mu m \)

- Spectral width \( S(\omega) \)

- Coherence Length:

\[ l_c = C \cdot \tau_c \quad (5.23) \]

- Rule of thumb:

\[ l_c = \frac{\lambda_0^2}{\Delta \lambda} \quad (5.24) \]
5.8 Optical Domain Refractometry

- Low-Coherence interferometry (interf. with broad band light).
- Sources: SLD, LED, white light, femtosecond laser, etc.

→ consider a transparent, layered structure under investigation.
5.8 Optical Domain Refractometry

- Scanning M, we retrieve

\[ R \]

⇒ The interface are resolved
⇒ Reflectivity give info about refractive index
⇒ \( L_{1,2,3,4} \) determine position of interfaces.

- ODR:
  - Successful for quantifying lasers in waveguides, fiber optics, etc.
5.9 Optical Coherence Tomography (OCT)

- Optical technique capable of rendering 3D images from think biological samples.
- Penetrates 1-2 mm deep in tissue.

=> Typically implemented in optical fiber configuration.

![Diagram of OCT setup]

- Tissue = continuous superposition of interfaces.
- Scanning M, a depth-resolved reflectivity signal is retrieved.

=> can resolve regions inside tissue (e.g. Tumors).

1991, Fujimoto’s group, MIT

Scan beam x-y
Scan M -> z
5.9 Optical Coherence Tomography (OCT)

- If mirror is swept at constant speed \( v \):
  \[ z = vt \]
  \[ \Rightarrow \text{Phase delay: } \phi = 2kz = 2kvt \] (2 means back and forth)
  \[ \Rightarrow \text{Frequency shift: } \Delta \omega = \phi = 2kvt \rightarrow \text{Doppler Shift} \]  

- The detector is recording a high-frequency signal \( \rightarrow \text{Low-noise} \)
- Dynamic range can easily reach 10 orders of magnitude! i.e. can record reflectivities from 1 to 1/10 billion! (100 dB)
- **Various Technological Improvements:**
  - Spectral domain OCT: instead of scanning \( M \), measure \( S(\omega) \rightarrow \Gamma(\tau) \)
  - Galvo-scanning group delay - fast
  - Swept source OCT
5.9 Optical Coherence Tomography (OCT)

- Various Technological Improvements:
  - Spectral encoding - instead of scanning on x, illuminate with $\lambda(x)$
  - Spectroscopic OCT - trade z-resolution for $S(\omega)$ information

- Since 1991, ~5,000 OCT papers published.
- Clinically applied in: Oftalmology
- Research stage: Dermatology, cardio, breast cancer, etc
- Recently combined with SHG, molecular imaging
5.10 Dispersion Effects on Temporal Coherence

- What happens if on one area of the Michelson interferometer, there is extra material (eg. Glass)?

![Diagram of a Michelson interferometer with additional material (glass) on one arm, labeled as 1 pass.](image-url)
5.10 Dispersion Effects on Temporal Coherence

- How does broad band fields propagate through dispersive materials (such as glass)? Think pulses!

- The phase delay through a transparent material:
  \[
  \phi(\omega) = k(\omega) \cdot d = k_0 \cdot d \cdot n(\omega)
  \]  
  \[(5.26)\]

\[\begin{align*}
  n(\omega) & \quad \text{Incident spectrum} \\
  k(\omega) = k_0 \cdot n(\omega) & \quad \text{Above resonance} \\
  S(\omega) & \quad \text{Below resonance}
\end{align*}\]
5.10 Dispersion Effects on Temporal Coherence

\[ n(\omega) = n(\omega_0) + \frac{dn}{d\omega} \mid_{\omega_0} (\omega - \omega_0) + \frac{1}{2} \frac{\partial^2 n}{\partial \omega^2} (\omega - \omega_0)^2 + \ldots \]

- **Note:**
  \[ n(\omega_0) \Rightarrow \phi_0 = n(\omega_0) \cdot k_0 \cdot d = \text{constant} \rightarrow \text{not important} \]
  \[ \phi(\omega) = \phi_0 + \frac{dk}{d\omega} \cdot d \cdot (\omega - \omega_0) + \frac{1}{2} \frac{d^2 k}{d\omega^2} \cdot d \cdot (\omega - \omega_0)^2 \ldots \]

- **Definitions:**
  - \[ \frac{dk}{d\omega} = v = \text{group velocity} \quad (5.27) \]
  - \[ \frac{d^2 k}{d\omega^2} = \beta_2 = \text{group velocity dispersion (GVD)} \]
  - Different colors have different group velocities
5.10 Dispersion Effects on Temporal Coherence

- E.g. Pulse: blue red
  \[ v_{\text{blue}} < v_{\text{red}} \]

- Riding with pulse \((v=0)\) → parabolic phase:

- So \( \phi(\omega) = \frac{1}{2} \beta_2 \cdot \omega^2 \) \hspace{1cm} (5.28)

- Cross-Spectral density:
  \[ W(\omega) = \langle E_1(\omega) \cdot E_2^*(\omega) \rangle \] \hspace{1cm} (5.29)

- Then, the cross-correlation function is

  \[ \Gamma_{12}(\tau) = \int_{0}^{\infty} W(\omega) e^{-i\omega \tau} d\omega \] \hspace{1cm} (5.30)
5.10 Dispersion Effects on Temporal Coherence

- \( \Gamma_{12}(\tau) = \int_{0}^{\infty} W(\omega) e^{-i\omega \tau} d\omega \) is the generalization of (5.16) i.e generalized Wiener-Kinntilin Theorem.

- For the “unbalanced” Michealson, the cross-spectral density is

\[
W(\omega) = E_1(\omega) \cdot E_2^*(\omega) = |E_1||E_2^*| e^{i \frac{1}{2} \beta_2 \omega^2} = S(\omega) e^{i \frac{1}{2} \beta_2 \omega^2}
\]

(5.32)

- The cross-correlation function:

\[
\Gamma_{12}(\tau) = \mathcal{F}[W(\omega)] = \mathcal{F}[S(\omega) e^{-i \frac{1}{2} \beta_2 \omega^2}]
\]

(5.33)

- Remember convolution theorem:

\[
\Gamma(\tau) = \mathcal{F}[S(\omega)] \bigodot \mathcal{F}[e^{-i \frac{1}{2} \beta_2 \omega^2}] = \Gamma_0(\tau) \bigodot h(\tau)
\]

(5.34)
5.10 Dispersion Effects on Temporal Coherence

- \[ h(\tau) = \mathcal{F}[e^{\frac{i}{2}\beta\omega^2}] \]

- Useful Fourier Transform relationship for Gauss functions:

\[
e^{-b \omega^2} \xrightarrow{\mathcal{F}} \frac{1}{\sqrt{2b}} e^{-\frac{t^2}{4b}}
\]

\[
h(\tau) = \frac{e^{\frac{i}{2}\beta\tau^2}}{\sqrt{i\beta}} \phi(\tau) \sim \tau^2 \text{ also parabolic}
\]

\[
\text{Diagram showing evolution of function with dispersion}
\]
5.10 Dispersion Effects on Temporal Coherence

- The coherence time is increased
- Frequency is “chirped”

- So, in OCT, is important to balance the interferometer=> minimum coherence length
- \( l_c \) gives the depth resolution