Chapter 9 - Polarization

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Polarization

- The x-y components of the field can be expressed via a 2D "Jones" vector:

$$J = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} A_x e^{i\phi_x} \\ A_y e^{i\phi_y} \end{pmatrix} \tag{9.1}$$

- Time variation:

$$\begin{aligned} E_x &= A_x \cos(\omega t + \phi_x) \\ E_y &= A_y \cos(\omega t + \phi_y) \end{aligned} \tag{9.2}$$

Polarization

- Eliminate time:

$$E_x = A_x\left[\cos(\omega t)\cos(\phi_x) - \sin(\omega t)\sin(\phi_x)\right]$$
$$E_y = A_y\left[\sin(\omega t)\sin(\phi_y) + \cos(\omega t)\cos(\phi_y)\right] \qquad (9.3)$$

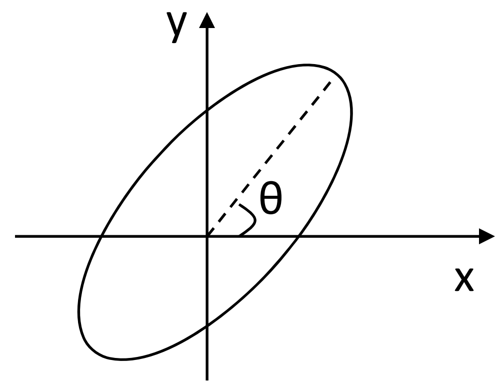
- Take as exercise to prove that the trajectory of the E-field is:

$$\frac{{E_x}^2}{{A_x}^2} + \frac{{E_y}^2}{{A_y}^2} - \frac{2E_xE_y}{A_xA_y}\cos\Delta\phi = \sin^2\Delta\phi \qquad (9.4)$$

where $\Delta\phi = \phi_y - \phi_x$

Polarization

- This is an ellipse:



- In general, it's rotated by angle θ
- This is what we would "see" looking straight at the incoming beam
- There are few particular cases:

a) Linear polarization: $\Delta\phi = 0, \pi$

$$\Rightarrow \left(\frac{E_x}{A_x} \pm \frac{E_y}{A_y} \right)^2 = 0 \Rightarrow \underline{\text{line}}$$

Polarization

b) Circular polarization: $\begin{cases} \Delta\phi = \pm \pi/2 \\ A_x = A_y \end{cases}$

$\pi/2 \rightarrow$ clockwise

$-\pi/2 \rightarrow$ clockwise

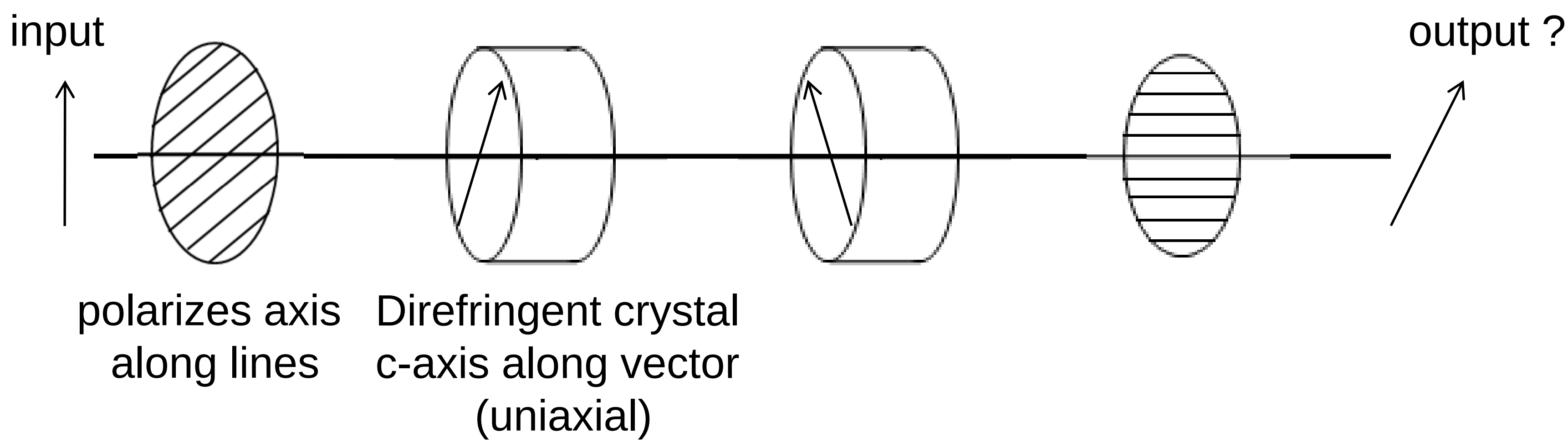
$$\Rightarrow E_x{}^2 + E_y{}^2 = A^2$$

c) Straight ellipse: $\begin{cases} \Delta\phi = \pm \pi/2 \\ A_x \neq A_y \end{cases}$

Jones Calculus for Birefringent Optical Systems

Introduction

- ABCD matrices told us about amplitude distribution on propagation
- Jones "calculus" deals with polarization changes for polarization sensitive optical elements



Introduction

- Jones "calculus" developed around 1940
- Based on 2x2 matrices (similat to ABCD)
    - Need 2 to describe Z polarization
    - Light travels 1 of 2 tranverse normal modes
- Birefringent crystals play role because two modes travel at different phase velocities
- Uniaxials simple – c-axis in plate surface

Retardation Plates (Waveplates)

- Change polarization state
- Assumption: no reflections at surface of elements (can use anti-reflection cratings)
- Use Jones matrices discussed before

$$\vec{E} = \frac{1}{2}\vec{V}\underbrace{E(x,y,z)}_{\text{Slow variation O.K.}}e^{j(\omega t - kz)} + cc$$

$\vec{V}$ describes polarization

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix};\ V_x^{\ 2} + V_y^{\ 2} = 1$$

Retardation Plates (Waveplates)

- E.g.
  - x-polarized $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

    $$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -j \end{pmatrix}$$

    $e^{-j\pi/2}$

  - y-polarized $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

    $$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ j \end{pmatrix}$$

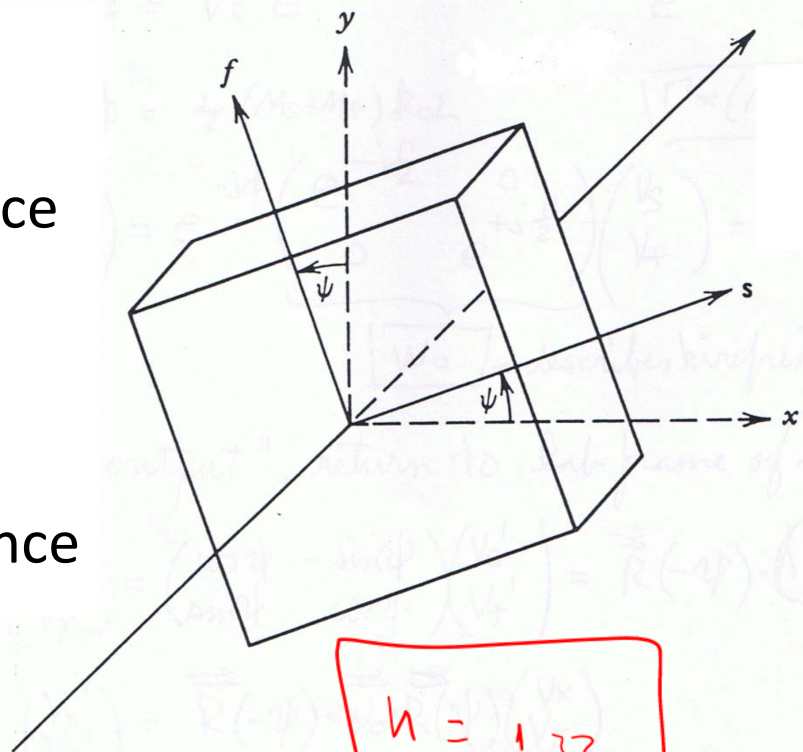
Retardation Plates (Waveplates)

$n^2 - 1$

$\bar{P} = \varepsilon_0 \bar{\bar{\chi}} \bar{E}$

x,y,z
laboratory
frame of reference

f,s,z
crystal
frame of reference



$n = 1.37$

f – “fast” axis
s – “slow” axis

$$\boxed{n_s > n_f}$$

$\bar{E}$

P_z

$$\bar{\bar{n}} = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ & \ddots & \\ & & n_{33} \end{pmatrix}$$

$\bar\nabla \times \bar E$

$\bar\nabla \times \bar H$

$\bar\nabla \cdot \bar D(\bar r,\omega) = \rho \qquad\qquad i\,\underbrace{\bar k \cdot \bar D}(k,\omega) = \cancel{\tilde\rho(k)}$

$\bar\nabla \cdot \bar B = 0 \qquad\qquad i\,\bar k \cdot \bar B = 0$

$$\boxed{\bar D = \bar\varepsilon\,\bar E}$$

$\varepsilon_0 E$

$k \perp D \longrightarrow \bar k \perp \bar E$

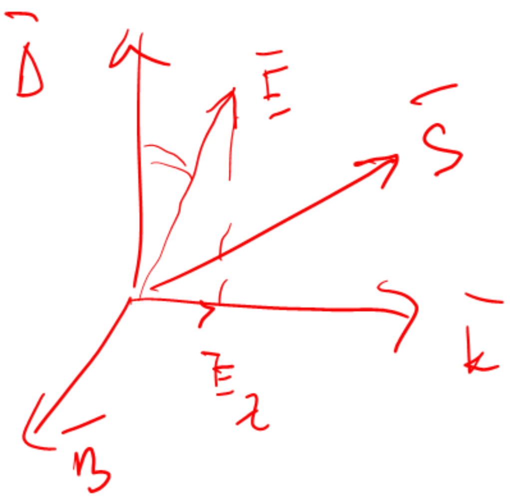
$\mu_0 H$

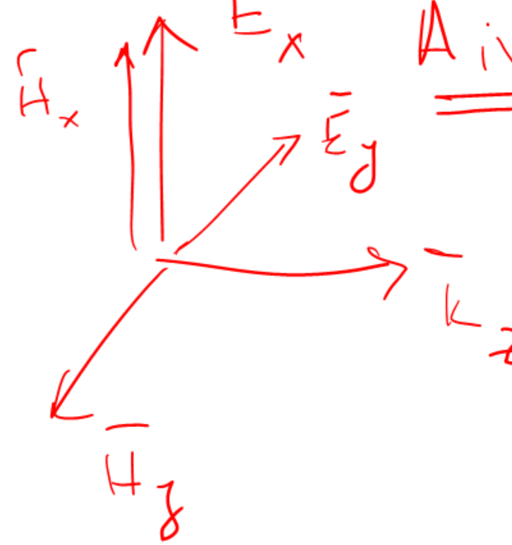
$\bar k \perp \bar B \Rightarrow \bar k \perp \bar H$

Air

K

$\bar k \perp \bar D \,;\; \bar k \perp \bar B$





$\bar S = \bar E \times \bar H$

Retardation Plates (Waveplates)

- From before

$$\underbrace{\begin{pmatrix} V_s \\ V_f \end{pmatrix}}_{\text{just inside the crystal}} = \underbrace{\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}}_{\boxed{R(\psi)}} \underbrace{\begin{pmatrix} V_x \\ V_y \end{pmatrix}}_{\text{just outside the crystal}}$$

- V_s & V_f propagate independently in the crystal
- After a distance L: $V_s' = V_s e^{-jn_s k_0 L}$

$$V_f' = V_f e^{-jn_f k_0 L}$$

Rewrite: $$V_s' = V_s e^{-j\frac{1}{2}(n_s + n_f)k_0 L} e^{-j\frac{1}{2}(n_s - n_f)k_0 L}$$

$$V_f' = V_f e^{-j\frac{1}{2}(n_s + n_f)k_0 L} e^{+j\frac{1}{2}(n_s - n_f)k_0 L}$$

$(\ ) \begin{pmatrix} V_s' \\ V_f' \end{pmatrix} = \begin{pmatrix} e^{-j(a)} \\ e^{+j(} \end{pmatrix}$

Retardation Plates (Waveplates)

- Define $\phi = \frac{1}{2}(n_s + n_f)k_0 L$ $\qquad \Gamma = (n_s - n_f)k_0 L$

$$\begin{pmatrix} V_s' \\ V_f' \end{pmatrix} = e^{-j\phi} \underbrace{\begin{pmatrix} e^{-j\frac{\Gamma}{2}} & 0 \\ 0 & e^{+j\frac{\Gamma}{2}} \end{pmatrix}}_{\boxed{W_0}} \begin{pmatrix} V_s \\ V_f \end{pmatrix} = e^{-j\phi}\,\overline{\overline{W_0}} \begin{pmatrix} V_s \\ V_f \end{pmatrix}$$

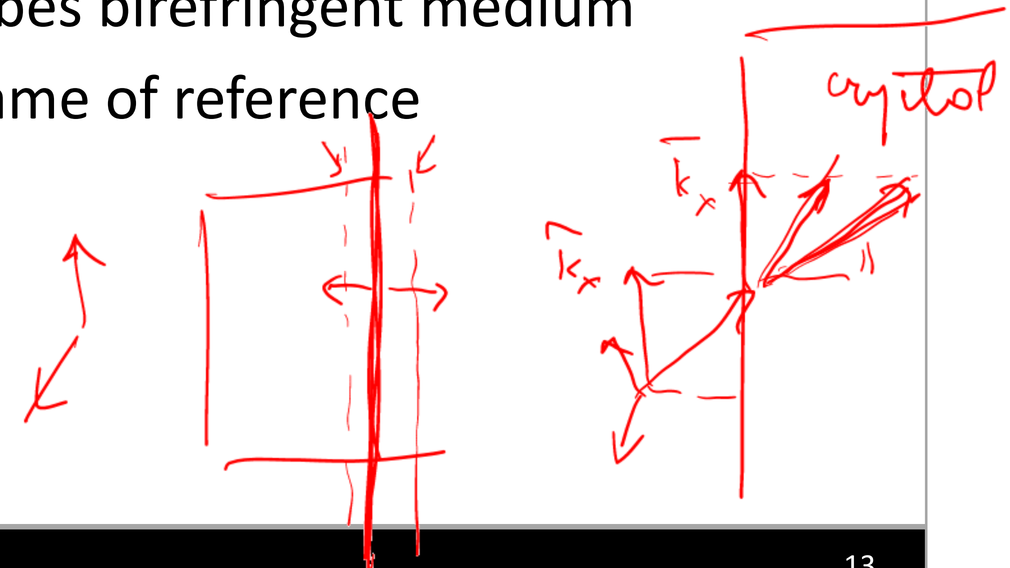
$|\bar{k}| = n k_0$

$\boxed{W_0}$ - describes birefringent medium

- To get "output", return to lab frame of reference

$s: \quad k_x^2 + k_y^2 + k_z^2 = n_s^2 k_0^2$

$f: \quad (\qquad\qquad) = n_f^2 k_0^2$



Retardation Plates (Waveplates)

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} V_s' \\ V_f' \end{pmatrix} = \overline{\overline{R}}(-\psi) \begin{pmatrix} V_s' \\ V_f' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \overline{\overline{R}}(-\psi) \cdot \overline{\overline{W_0}} \cdot \overline{\overline{R}}(\psi) \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

$\longleftarrow$ order of doing products of matrices

- <u>Note</u>: each matrix is unitary $\left\| R(-\psi) = 1 \right\|$

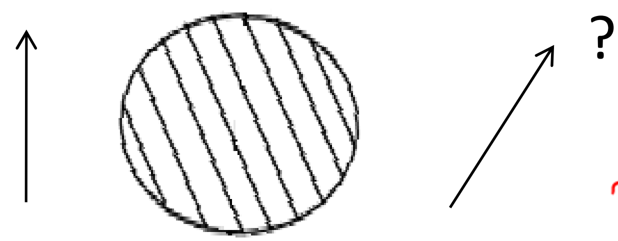
  → Polarization remain orthogonal

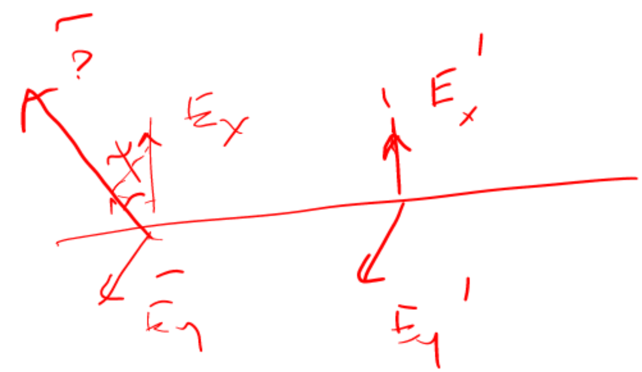
Polarizers

x-axis $P_x = e^{-j\phi_p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

y-axis $P_y = e^{-j\phi_p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

ϕ_p - phase change through polarizer





$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \overline{\overline{R}}(-\psi)\cdot\overline{\overline{P_0}}\cdot\overline{\overline{R}}(\psi)\begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

Half (1/2) Wave Plate

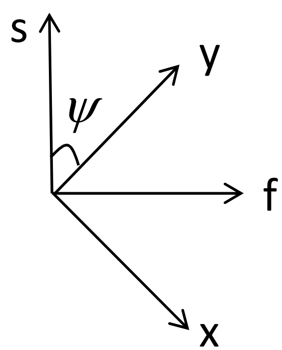
$$\boxed{\Gamma = \pi} \Rightarrow L = \frac{\lambda}{2}(n_s - n_f)$$

$$\Rightarrow e^{\pm j\frac{\pi}{2}} \Rightarrow \pm j$$

$e^{-i\frac{\Gamma}{2}}$

$e^{-i\frac{\pi}{2}}$

$= -i$



$$\overline{\overline{W}} = \overline{\overline{R}}(-\psi)\cdot\overline{\overline{W_0}}\cdot\overline{\overline{R}}(\psi)$$

$$= \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} -j & 0 \\ 0 & j \end{pmatrix}\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}$$

$$= j\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} -\cos\psi & -\sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}$$

$$= -j\begin{pmatrix} \cos^2\psi - \sin^2\psi & 2\sin\psi\cos\psi \\ 2\sin\psi\cos\psi & \sin^2\psi - \cos^2\psi \end{pmatrix}$$

½ Wave Plate

$$\boxed{\overline{\overline{W}} = -j\begin{pmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{pmatrix}}$$

→ rotates a plane of Polarization by 2 x Ψ

angle between lab and crystal frames

- Another important factor is the transmission through the element

$$T = \frac{|E'|^2}{|E|^2} = \frac{|V_x'|^2 + |V_y'|^2}{|V_x|^2 + |V_y|^2}$$

½ Wave Plate

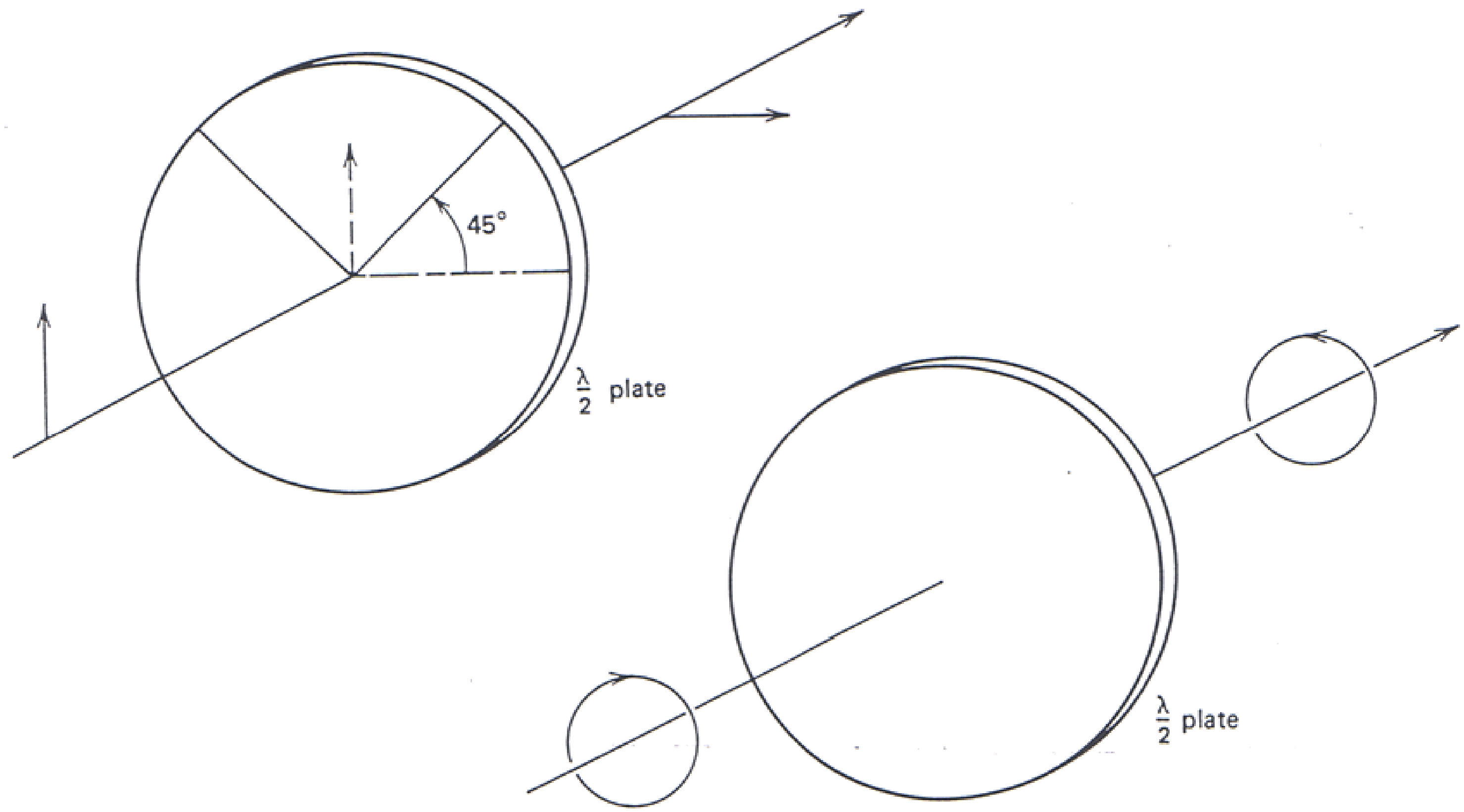
- E.g. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ input

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = -j \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -j \begin{pmatrix} \cos 2\psi \\ \sin 2\psi \end{pmatrix}_{if\,\psi = 45^o} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Just get net polarization rotation

$$\cos^2 2\psi + \sin^2 2\psi = 1 \Rightarrow \boxed{T = 1}$$

½ Wave Plate



½ Wave Plate

- E.g. $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ j\end{pmatrix}$ input

$$\begin{pmatrix}V_x'\\ V_y'\end{pmatrix}=\begin{pmatrix}\cos 2\psi & \sin 2\psi\\ \sin 2\psi & -\cos 2\psi\end{pmatrix}\begin{pmatrix}1\\ j\end{pmatrix}\frac{1}{\sqrt{2}}=-\frac{j}{\sqrt{2}}\begin{pmatrix}\cos 2\psi + j\sin 2\psi\\ \sin 2\psi - j\cos 2\psi\end{pmatrix}$$

$$T=\frac{1}{2}(\cos 2\psi + j\sin 2\psi)(\cos 2\psi - j\sin 2\psi)+(\sin 2\psi - j\cos 2\psi)(\sin 2\psi + j\cos 2\psi)$$

$$=\frac{1}{2}(\cos^2 2\psi+\sin^2 2\psi+\cos^2 2\psi+\sin^2 2\psi)=1$$

- Is the output still circulary polarized?
- Depends on Ψ

½ Wave Plate

- Choose $\psi = \frac{\pi}{4}$

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = -\frac{j}{\sqrt{2}}\begin{pmatrix} j \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -j \end{pmatrix}$$

(1) still circulary polarized

(2) reverses sense of rotation

(3) for $\psi \neq \frac{\pi}{4}$, get elliptical polarization!

¼ Wave Plate

$$\boxed{\Gamma = \frac{\pi}{2}} \Rightarrow L = \frac{\lambda}{4}(n_s - n_f)$$

$$\Rightarrow e^{\pm j\frac{\pi}{4}} = \cos\frac{\pi}{4} \pm j\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1 \pm j)$$

Assume Ψ= 45^0:

$$\overline{\overline{W}} = \overline{\overline{R}}(-45^o)\cdot\overline{\overline{W_0}}\left(\frac{\pi}{2}\right)\cdot\overline{\overline{R}}(45^o) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-j\frac{\pi}{4}} & 0 \\ 0 & e^{j\frac{\pi}{4}} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}e^{-j\frac{\pi}{4}}\begin{pmatrix} 1+j & 1-j \\ 1-j & 1+j \end{pmatrix}$$

¼ Wave Plate

$$e^{i\pi/4} \leftarrow \begin{bmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix}$$

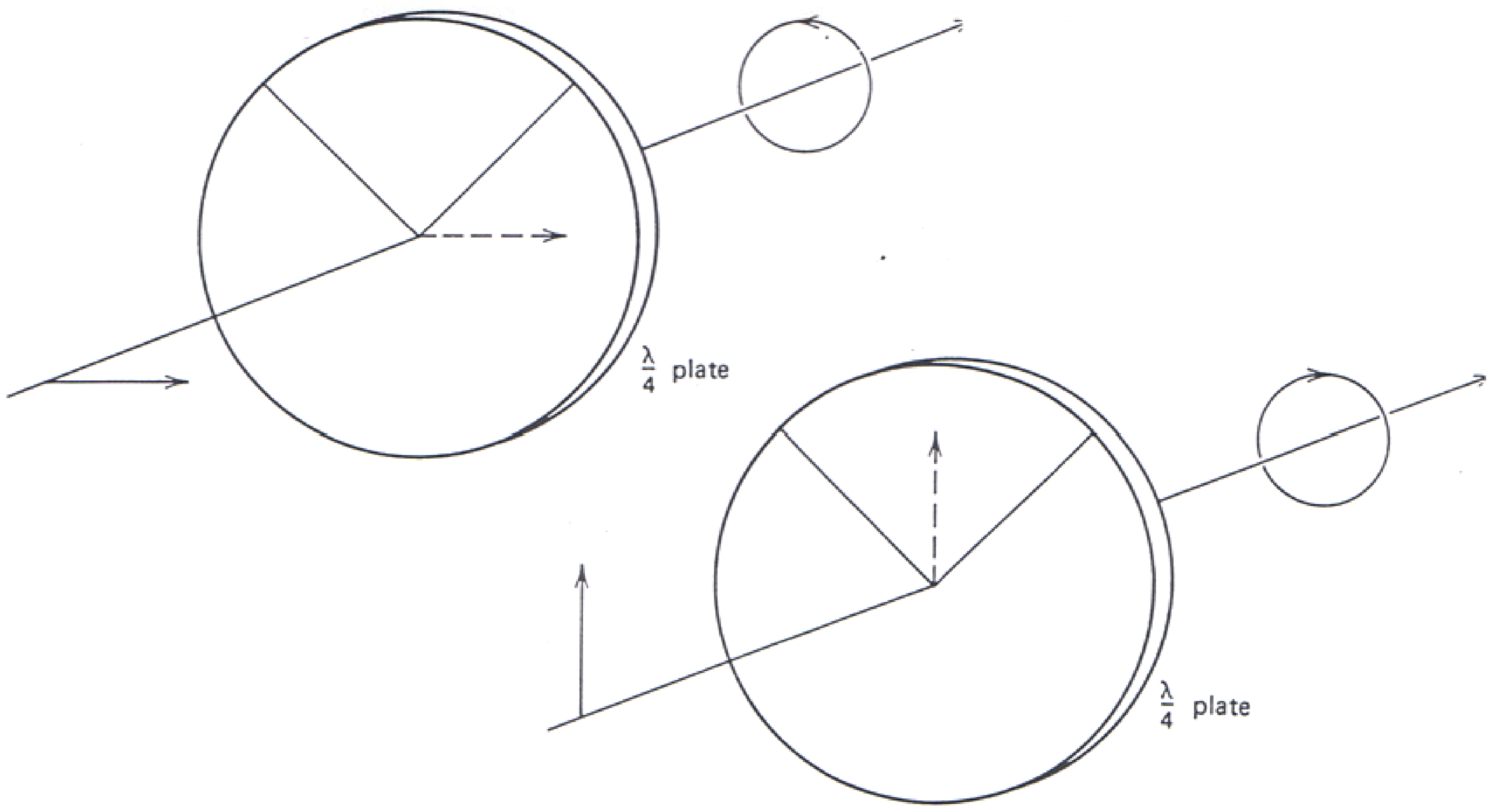
- E.g. $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} V_x{}' \\ V_y{}' \end{pmatrix} = \dfrac{e^{-j\frac{\pi}{4}}}{2} \begin{pmatrix} 1+j & 1-j \\ 1-j & 1+j \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \dfrac{e^{-j\frac{\pi}{4}}}{2} \begin{pmatrix} 1+j \\ 1-j \end{pmatrix}$

$$= \frac{1+j}{2} e^{-j\frac{\pi}{4}} \begin{pmatrix} 1 \\ -j \end{pmatrix} \qquad \text{RCP}$$

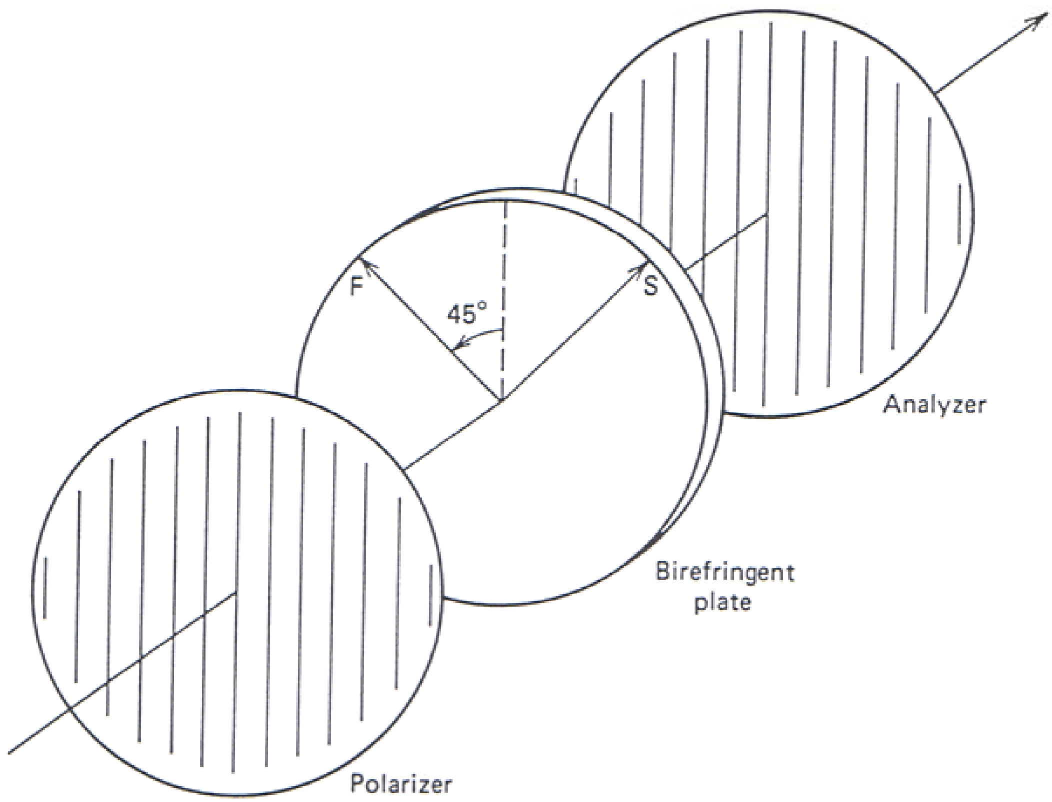
- Eg. $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 1 \\ -j \end{pmatrix}; \begin{pmatrix} V_x{}' \\ V_y{}' \end{pmatrix} = \dfrac{e^{-j\frac{\pi}{4}}}{2} \begin{pmatrix} 1+j & 1-j \\ 1-j & 1+j \end{pmatrix} \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -j \end{pmatrix} = \dfrac{e^{-j\frac{\pi}{4}}}{2\sqrt{2}} \begin{pmatrix} 0 \\ 2-2j \end{pmatrix}$

$$= \frac{1-j}{\sqrt{2}} e^{-j\frac{\pi}{4}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{y-polarized}$$

¼ Wave Plate



Waveplates and Polarizers



Waveplates and Polarizers

$$W = R(-45^o)\cdot W_0\left(\Gamma\right)\cdot R(45^o) = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{-j\frac{\Gamma}{2}} & 0\\ 0 & e^{j\frac{\Gamma}{2}}\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

$$=\frac{2}{2}\begin{pmatrix}\cos\frac{\Gamma}{2} & -j\sin\frac{\Gamma}{2}\\ -j\sin\frac{\Gamma}{2} & \cos\frac{\Gamma}{2}\end{pmatrix}=\begin{pmatrix}\cos\frac{\Gamma}{2} & -j\sin\frac{\Gamma}{2}\\ -j\sin\frac{\Gamma}{2} & \cos\frac{\Gamma}{2}\end{pmatrix}$$

$$Px\cdot W\cdot Px = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\Gamma}{2} & -j\sin\frac{\Gamma}{2}\\ -j\sin\frac{\Gamma}{2} & \cos\frac{\Gamma}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\Gamma}{2} & 0\\ -j\sin\frac{\Gamma}{2} & 0\end{pmatrix}=\begin{pmatrix}\cos\frac{\Gamma}{2} & 0\\ 0 & 0\end{pmatrix}$$

Waveplates and Polarizers

- Case I: unpolarized light $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\frac{\Gamma}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\Gamma}{2} \\ 0 \end{pmatrix}$$

linearly polarized: obvious !!

$$T = \frac{|V_x'|^2 + |V_y'|^2}{|V_x|^2 + |V_y|^2} = \frac{1}{2}\cos^2\frac{\Gamma}{2}$$

½ lost at first polarizer!

Waveplates and Polarizers

- Case II: $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \begin{pmatrix} \cos\frac{\Gamma}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\Gamma}{2} \\ 0 \end{pmatrix}$$

$$T = \cos^2\frac{\Gamma}{2}$$

Waveplates and Polarizers

- Case III: $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -j \end{pmatrix}$

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\frac{\Gamma}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -j \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\frac{\Gamma}{2} \\ 0 \end{pmatrix}$$

$$T = \frac{1}{2}\cos^2\frac{\Gamma}{2}$$

- For ½ wave plates, $\Gamma = \pi \Rightarrow T = 0$

Birefringent Plate between Crossed Polarizers

Ψ= 45⁰:

$$W = R(-45^o)\cdot W_0(\Gamma)\cdot R(45^o) = \begin{pmatrix} \cos\frac{\Gamma}{2} & -j\sin\frac{\Gamma}{2} \\ -j\sin\frac{\Gamma}{2} & \cos\frac{\Gamma}{2} \end{pmatrix}$$

$$Py\cdot W\cdot Px = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\frac{\Gamma}{2} & -j\sin\frac{\Gamma}{2} \\ -j\sin\frac{\Gamma}{2} & \cos\frac{\Gamma}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -j\sin\frac{\Gamma}{2} & 0 \end{pmatrix}$$

Birefringent Plate between Crossed Polarizers

- <u>Case I</u>: unpolarized light $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 e^{i\phi_x} \\ 1 e^{i\phi_y} \end{pmatrix}$; $\langle(\phi_x - \phi_y)\rangle = 0$

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -j\sin\frac{\Gamma}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -j\sin\frac{\Gamma}{2} \end{pmatrix}$$

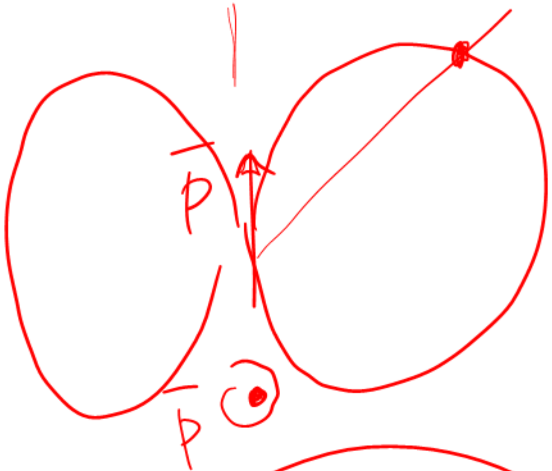
$$T = \frac{1}{2}\sin^2\frac{\Gamma}{2}$$

Birefringent Plate between Crossed Polarizers

- Case II: $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

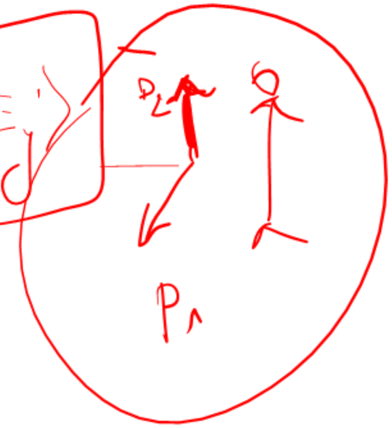
$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -j\sin\frac{\Gamma}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -j\sin\frac{\Gamma}{2} \end{pmatrix}$$

$$T = \sin^2\frac{\Gamma}{2}$$



$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \qquad \boxed{\Gamma_{ij} = \langle E_i E_j \rangle}$$

$$\langle U(t)\, U^*(t+\tau) \rangle$$



$$\langle \phi_L(t) \phi_R(t+\tau) \rangle = \delta(\tau)$$

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = |E_x| \begin{pmatrix} 1 \\ 1\,e^{i\pi/2} \end{pmatrix}$$