Chapter 6 – Fourier Optics

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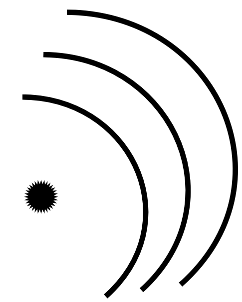
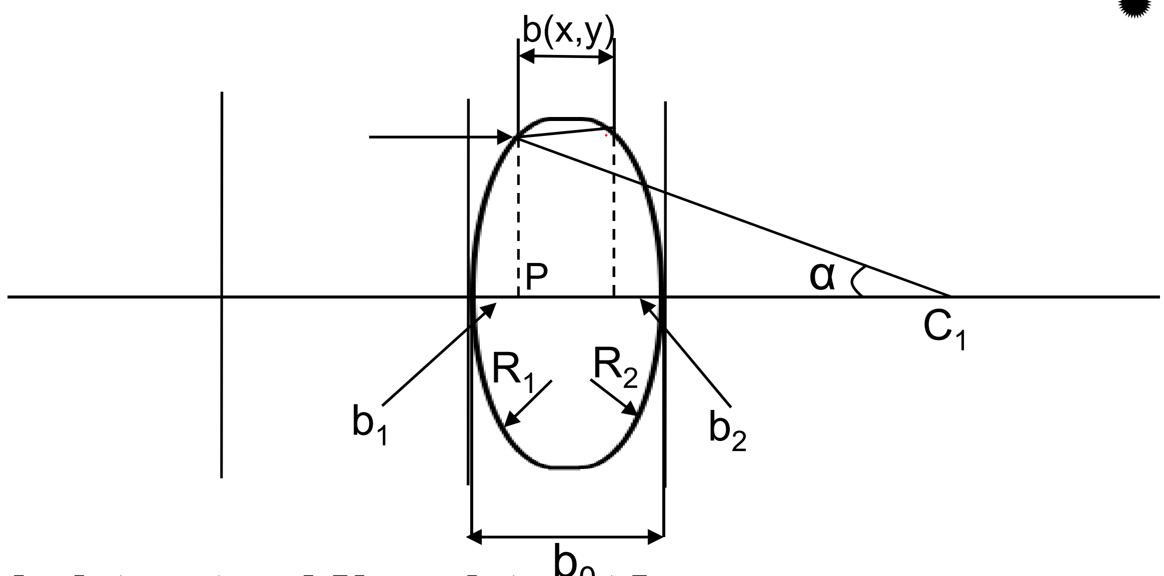
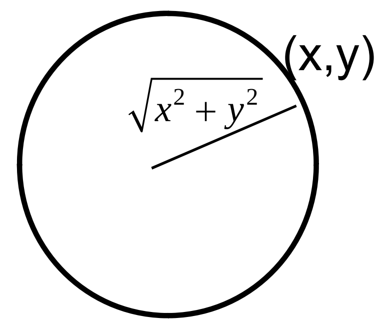
3.10 Lens as a phase transformer

- $E = E_o \cdot e^{i\phi}$ becomes important
- How is the wavefront changed by a lens?







$$\phi(x,y) = \underbrace{knb(x,y)}_{\text{glass}} + \underbrace{k[b_o - b(x,y)]}_{\text{air}} \qquad (3.32)$$

$$= kb_o + k(n-1)b(x,y)$$

3.10 Lens as a phase transformer

- Let's calculate b(x,y); assume small angles

$$b_1 = R_1 - (PC_1) =$$
$$= R_1 - \sqrt{R_1^{\ 2} - (\alpha R_1)^2} = R_1\left[1 - \sqrt{1-\alpha^2}\right]$$

- Taylor expansion: $\sqrt{1+x}\,|_{x\to o} \simeq 1 + \frac{x}{2}$

Small Angle Approx (Gaussian)

$$\rightarrow b_1 = R_1\left[1 - \left(1 - \frac{\alpha^2}{2}\right)\right] = R_1\frac{\alpha^2}{2} \qquad (3.33)$$

- $\alpha \simeq \tan\alpha = \dfrac{\sqrt{x^2+y^2}}{R_1}$

- So: $$\boxed{b_1(x,y) = \frac{x^2+y^2}{2R_1}} \qquad (3.34)$$

3.10 Lens as a phase transformer

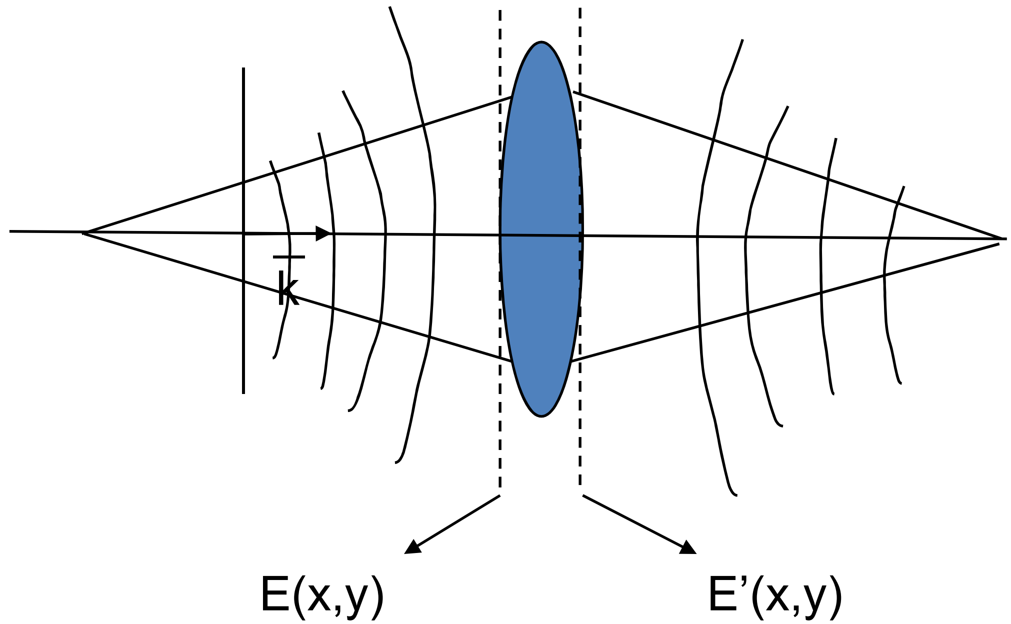
$$\rightarrow \; b(x,y) = b_o - b_1(x,y) - b_2(x,y) =$$
$$= b_o - \frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (3.35)$$

- This is the thickness approximation
- The phase ϕ becomes:

$$\phi(x,y) = \phi_o - k(n-1)b(x,y) =$$
$$= \phi_o - k\frac{x^2+y^2}{2}(n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \qquad (3.36)$$

- But we know: $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

3.10 Lens as a phase transformer



$$\rightarrow E'(x,y) = E(x,y)\cdot t_e(x,y) \qquad (3.37)$$

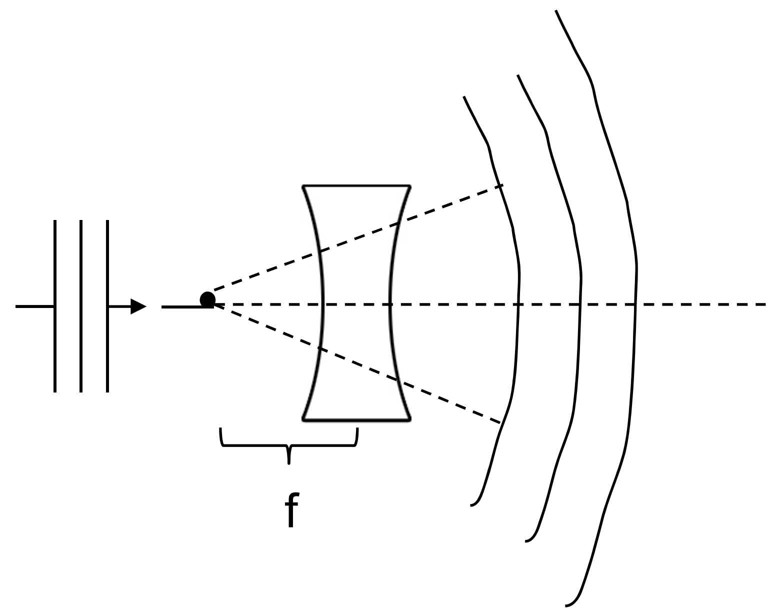
- The lens transformation is:

$$t_e = e^{i\phi} = e^{iknb_o}.e^{-i\frac{k}{2f}(x^2+y^2)} \qquad (3.38)$$

3.10 Lens as a phase transformer

- A lens transforms an incident plane wavefront into a parabolic shape
- Note: $f > 0$ covergent lens

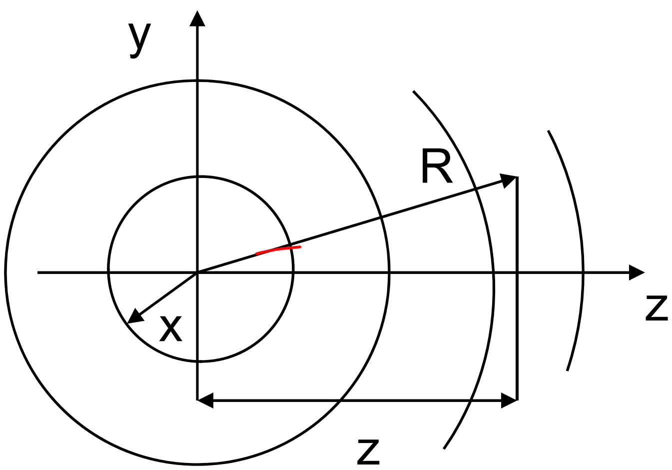
  $f < 0$ divergent



- So, if we know how to propagate through free space, then we can calculate field amplitude and phase through any imaging system

3.11 Huygens-Fresnel principle

- Spherical waves:
- Wavelet: $h = \dfrac{e^{ikR}}{R}$
- $R = \sqrt{x^2 + y^2 + z^2} = z\sqrt{1 + \dfrac{x^2 + y^2}{z^2}}$



- We are interested close to OA, i.e. small angles

$$\rightarrow \quad R \simeq z\left[1 + \frac{1}{2}\left(\frac{x^2 + y^2}{z^2}\right)\right] \qquad (3.39)$$

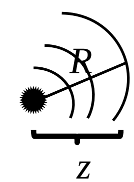
3.11 Huygens-Fresnel principle

- For amplitude $1/R \simeq 1/z$ is OK

$$e^{ikR}$$

$$R \simeq z$$

- For phase $kR \simeq kz\left[1+\frac{1}{z}\left(\frac{x^2+y^2}{z^2}\right)\right]$



→ The wavelet becomes:

$$h(x,y) \simeq \frac{e^{ikz}}{z} e^{i\frac{k(x^2+y^2)}{2z}} \quad (3.40a)$$

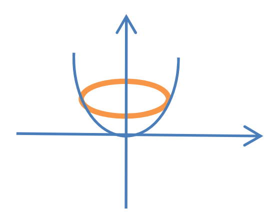
$$f(x,y) = x^2 + y^2$$

- Remember, for the lens we found:

$$f(x) = x^2$$

$$t_e(x,y) = e^{i\phi_o} e^{i\frac{k}{2f}(x^2+y^2)} \quad (3.40b)$$

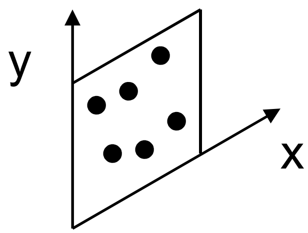
← Negative Lens



- Free space acts on the wavefront like a <u>divergent</u> lens (note "+" sign in phase)

3.11 Huygens-Fresnel principle

- At a given plane, a field is made of point sources

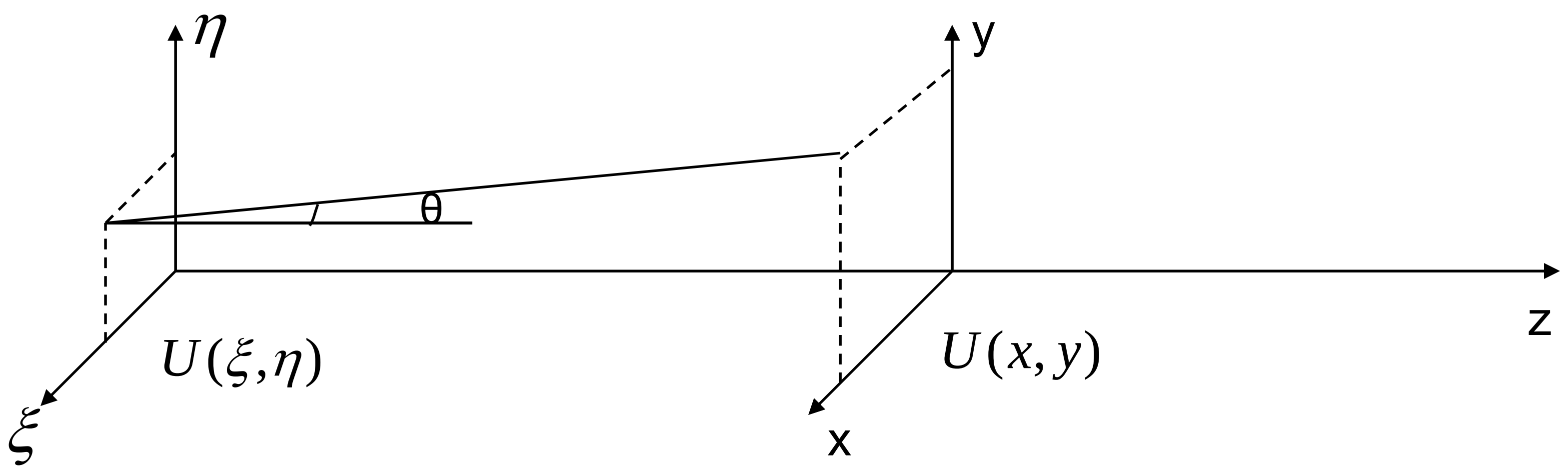


$$E(x,y)=\iint E(x',y')\delta(x-x')\delta(y-y')dx'dy'$$

- Eq 3.40 a-b represent the impulse response of the system (free space or lens)
- Recall linear systems (Chapter 2, page 12, Eq 2.16)
  - Final response (output) is the convolution of the input with the impulse response (or Greeen's function)
- Nice! Space or time signals work the same!

$$\int f(x')\delta(x_0-x')dx' => f(x_0)$$

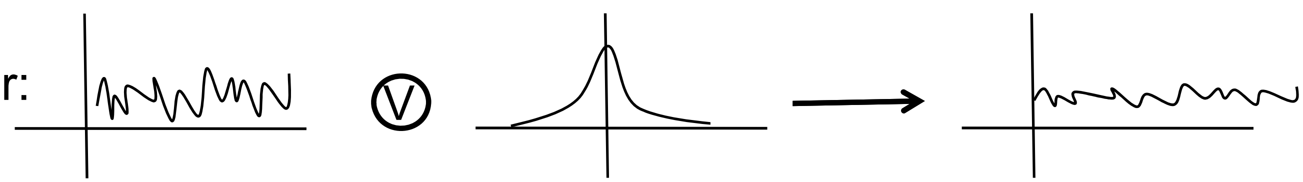
3.11 Huygens-Fresnel principle



$$U(x,y)=\iint U(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta$$

$$U(x,y)=\iint U(\xi,\eta)e^{\frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]}d\xi d\eta \qquad (3.41)$$

Remember:



3.11 Huygens-Fresnel principle

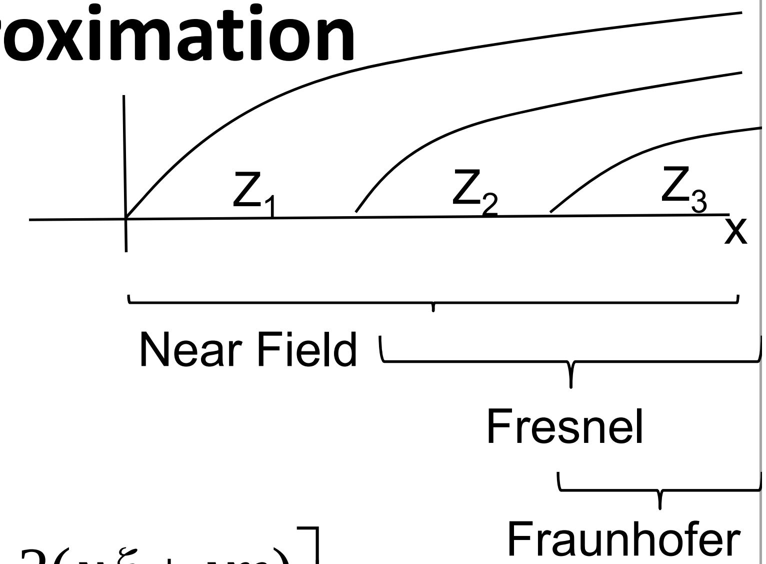
- Fresnel diffraction equation = convolution
- Fresnel diffraction equation is an approximation $\left(R = z\left[1+\frac{x^2+y^2}{2z^2}\right]\right)$ of Huygens principle (17th century)

$$U(x,y) = \frac{1}{i\lambda}\int\int U(\xi,\eta)\frac{e^{ikR(\xi,\eta)}}{R(\xi,\eta)}\cos\theta(\xi,\eta)d\xi d\eta \qquad (3.42)$$

- ! Fresnel is good enough for our pourpose
- Note: we don't care about constants A (no x-y dependence)

$$U(x,y) = \int\int U(\xi,\eta)e^{\frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]}d\xi d\eta$$

3.12 Fraunhofer Approximation



- One more approximation (far field)
- The phase factor in Fresnel is:

$$\phi(x,y) = \frac{k}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right] =$$

$$= \frac{k}{2z}\left[(x^2+y^2) + \underbrace{(\xi^2+\eta^2)}_{\approx 0} - 2(x\xi + y\eta)\right] \quad (3.43)$$

- If $z >> k(\xi^2+\eta^2)$, we obtain the Fraunhofer equation:

$$U(x,y) = A\iint_{-\infty}^{\infty} U(\xi,\eta) e^{-\frac{i2\pi}{\lambda z}(x\xi+y\eta)} d\xi d\eta \quad (3.44)$$

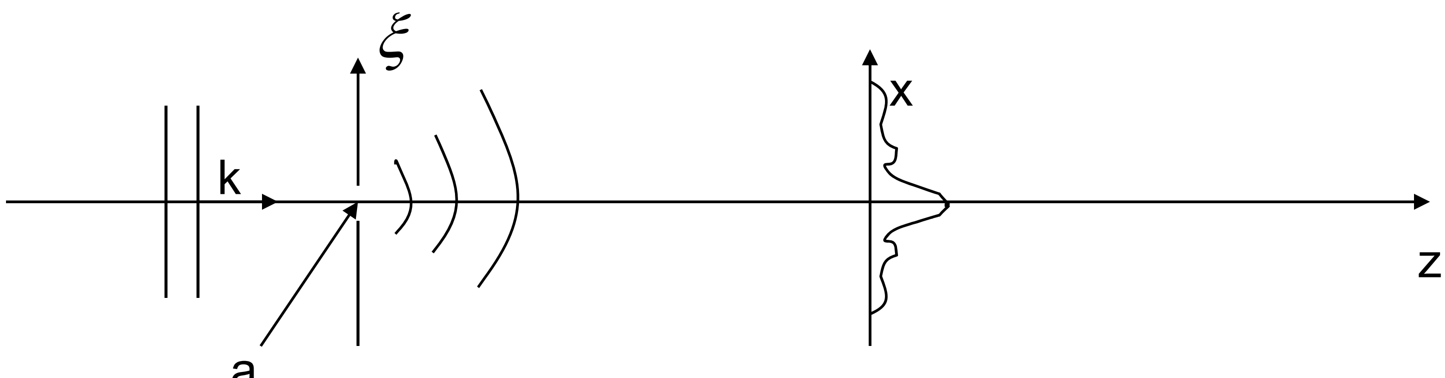
- Thus eq. 3.44 defines a Fourier transform
- Useful to calculate diffraction patterns !

3.12 Fraunhofer Approximation

- Let's define: $\begin{cases} f_x = \dfrac{x}{\lambda z} \\ f_y = \dfrac{y}{\lambda z} \end{cases}$

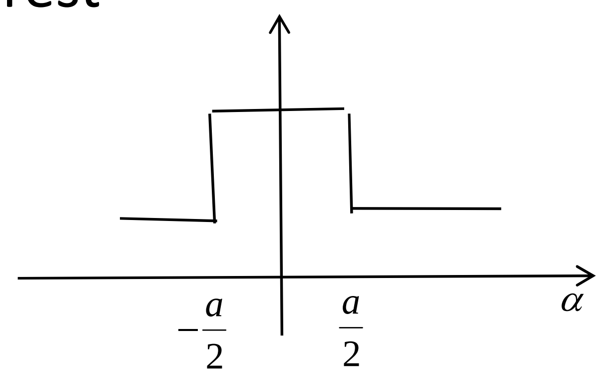
$$\rightarrow U(f_x, f_y) = \iint_{-\infty}^{\infty} U(\xi,\eta).e^{-i2\pi(\xi f_x + \eta f_y)} d\xi d\eta \qquad (3.45)$$

- Example: diffraction on a slit



3.12 Fraunhofer Approximation

- One dimensional: $U(x) = \Pi\left(\frac{x}{a}\right) =$ a, |x| < a/2; 0, rest



- The far-field is given by Fraunhofer eq:

$$U(f_x) = \int_{-\infty}^{\infty} U(x) e^{-i2\pi x f_x} dx =$$

$$= \Im\left[\Pi\left(\frac{x}{a}\right)\right] =$$

- Similarity Theorem + $\Im[\Pi(x)] = \sin c(f_x)$:

→ $U(f_x) = a \sin c(a f_x) =$

$$= a \frac{\sin(a f_x)}{a f_x} \qquad (3.46)$$

- $f_x = \frac{x}{\lambda z}$

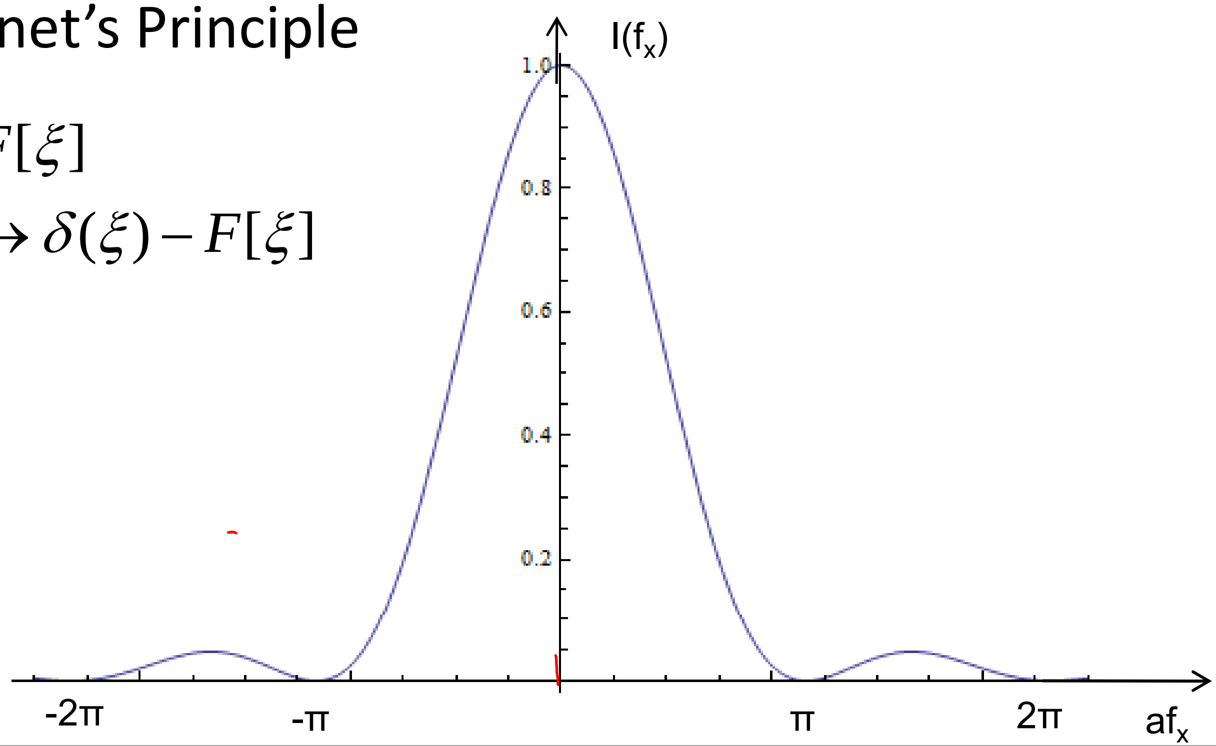
3.12 Fraunhofer Approximation

- Always measure intensity → the diffraction pattern is:

$$I(f_x) = |U(f_x)|^2 = a^2 \left[\frac{\sin(af_x)}{af_x} \right]^2 \qquad (3.48)$$

- Also Babinet's Principle

$$f(x) \to F[\xi]$$
$$1 - f(x) \to \delta(\xi) - F[\xi]$$

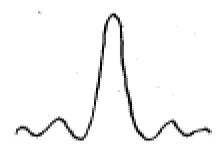


3.12 Fraunhofer Approximation

- <u>Note</u>: $\sin(af_x) = \sin\left(\frac{f_x}{\frac{1}{a}}\right)$ → $\frac{1}{a} =$ width of diffraction pattern

- narrow slit: a_1 →

- wide slit: a_2 →

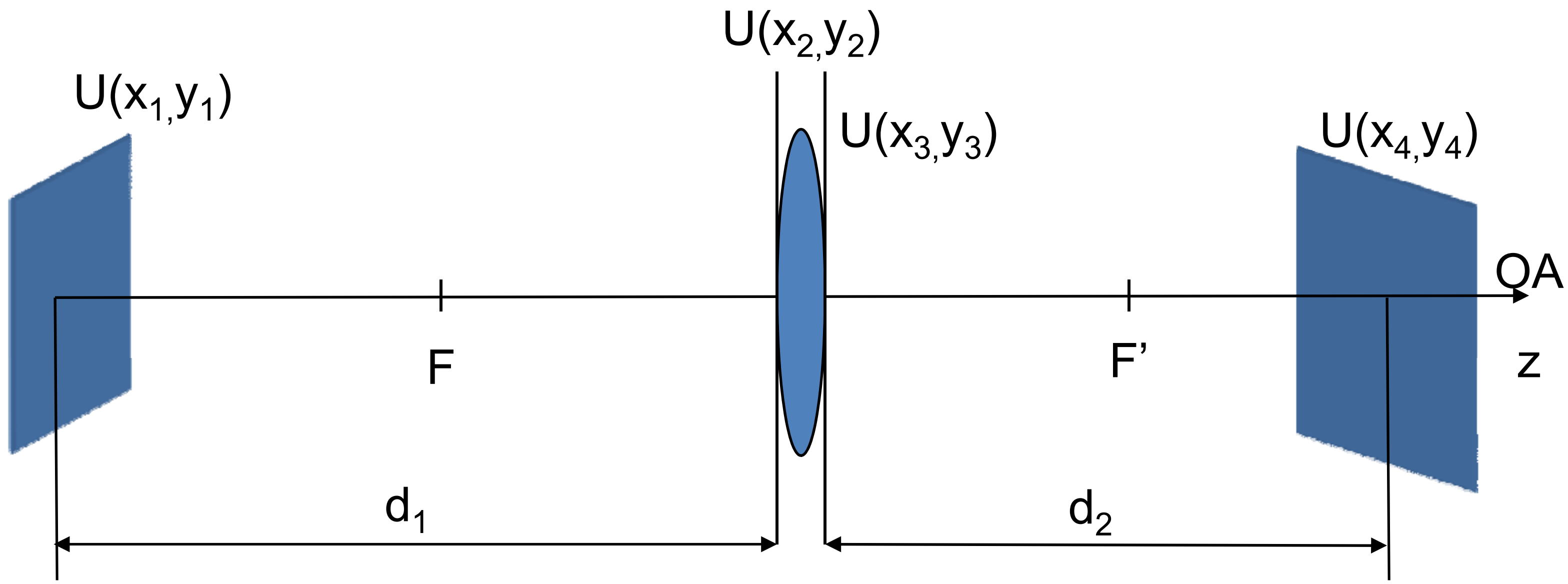


- <u>Similarity Theorem</u> ↔ uncertainty principle

Quiz:

What is the diffraction pattern from 2 slits of size a separated by d?

3.13 Fourier Properties of lenses



- Propagation:

$$U(x_1,y_1) \xrightarrow{\text{Fresnel}} U(x_2,y_2)$$

$$U(x_2,y_2) \xrightarrow{\text{Lens}} U(x_3,y_3)$$

$$U(x_3,y_3) \xrightarrow{\text{Fresnel}} U(x_4,y_4)$$

3.13 Fourier Properties of lenses

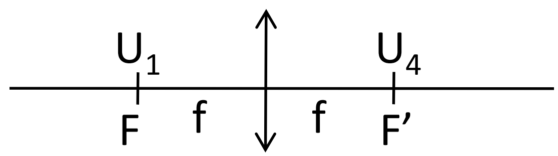
- $U(x_2, y_2) = A_{12} \iint U(x_1, y_1) e^{\frac{ik}{2d_1}\left[(x_2-x_1)^2+(y_2-y_1)^2\right]} dx_1 dy_1$
- $U(x_3, y_3) = A_{23} U(x_3, y_3) e^{-i\frac{k}{2f}\left[x_3^2+y_3^2\right]}$
- $U(x_4, y_4) = A_{34} \iint U(x_3, y_3) e^{\frac{ik}{2d_2}\left[(x_4-x_3)^2+(y_4-y_3)^2\right]} dx_3 dy_3$

(3.49)

- Combining Eqs (3.49) is a little messy, but there is a special case when eqs simplify → very useful

3.13 Fourier Properties of lenses

- If $d_1 = d_2 = f$



$$\begin{cases} U(x_4, y_4) = A_{41} \int \int_{-\infty}^{\infty} U_1(x_1, y_1) e^{-i2\pi(x_1 f_x + y_1 f_y)} dx_1 dy_1 \\ f_x = \frac{x_4}{\lambda f}; f_y = \frac{y_4}{\lambda f} \end{cases} \quad (3.50)$$

- Same eq as (3.45); now <u>z → f</u>
- Lenses work as Fourier transformers
  - Useful for <u>spatial filtering</u>

What is $\left|\Im[U]\right|^2$

$$\Im[\left|U\right|^2] =$$

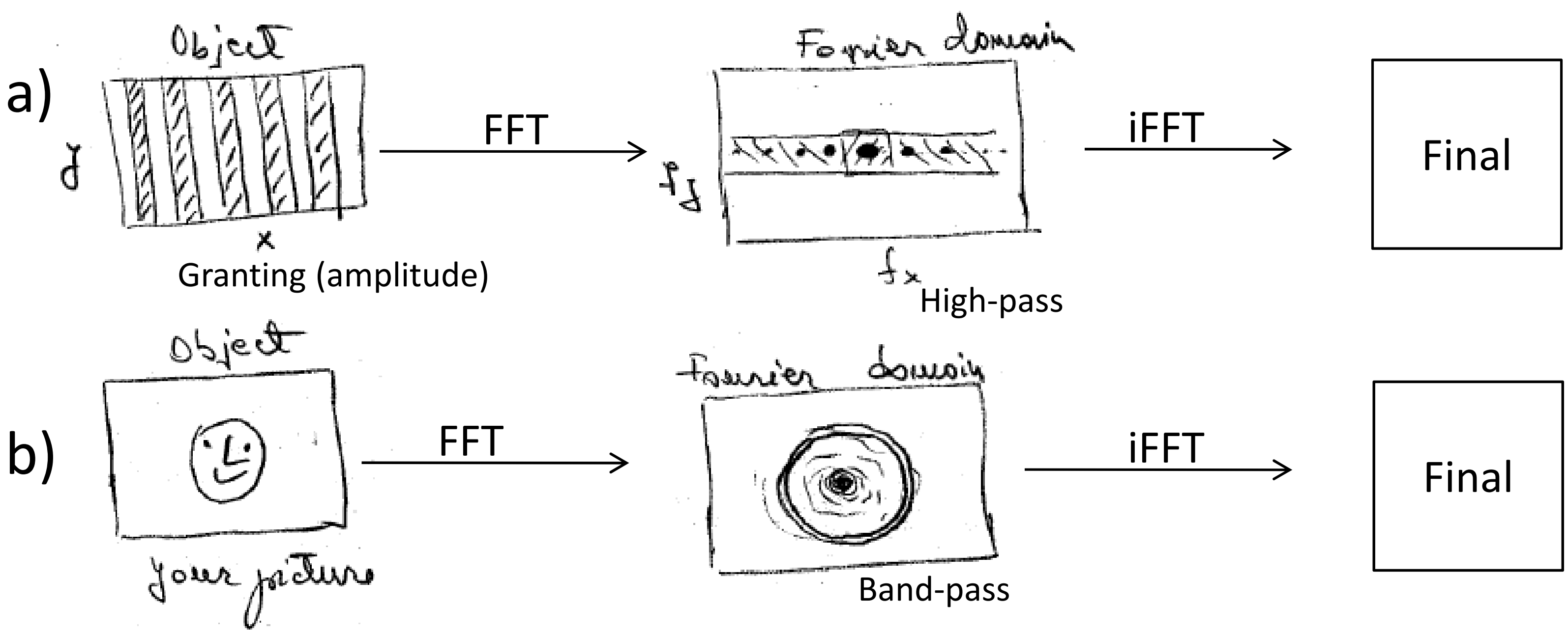
$$\Im[U \cdot U^*] =$$

$$\tilde{U} \otimes \tilde{U}^*$$

Autocorrelation

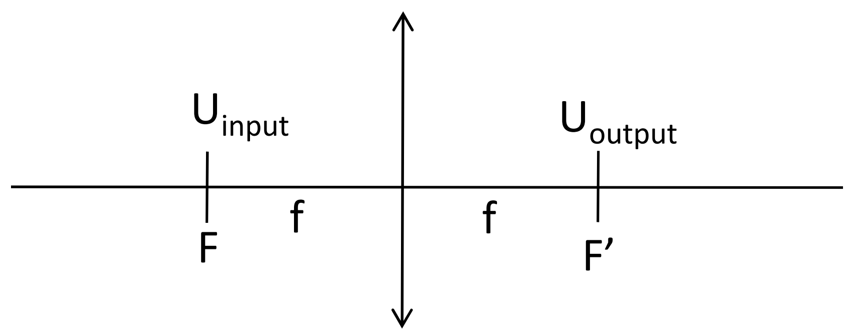
3.13 Fourier Properties of lenses

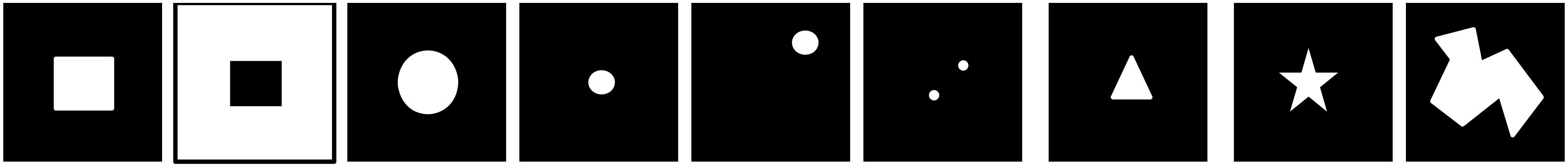
- Exercise: Use Matlab to FFT images (look up "fft2" in help)



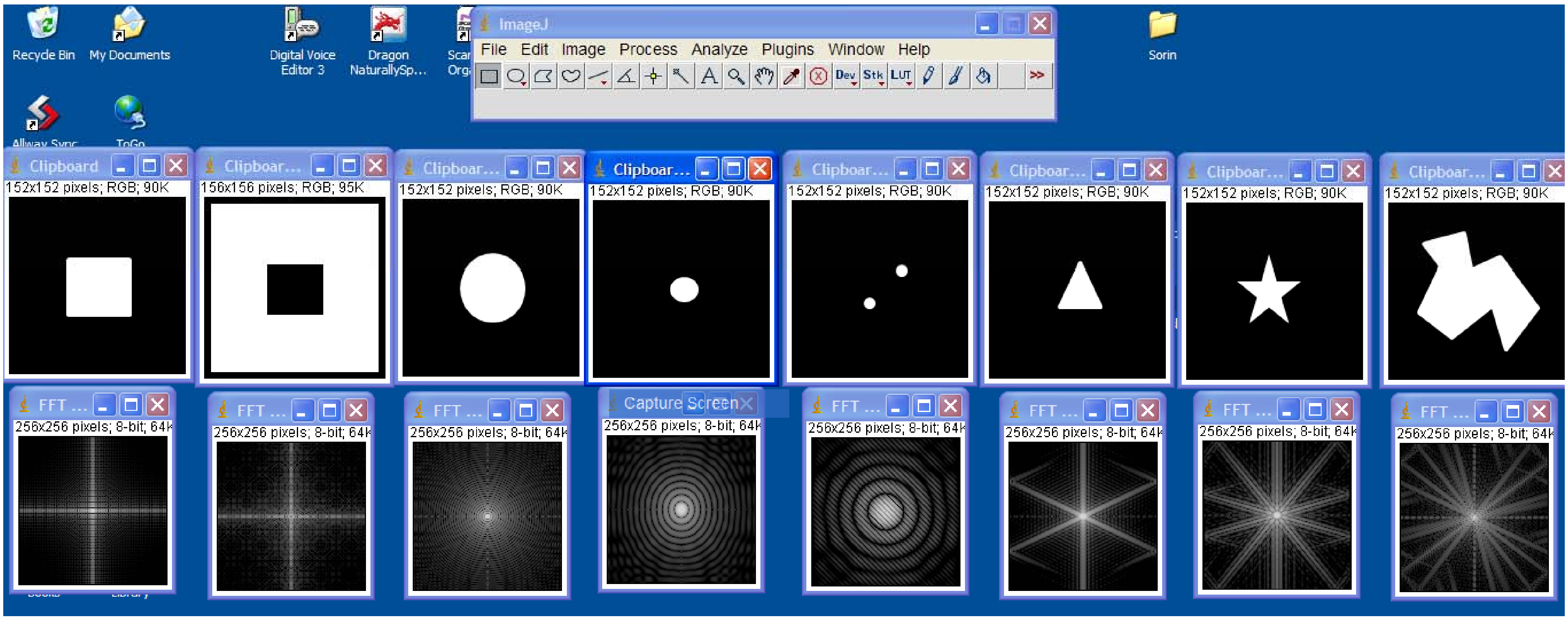
- Note the relationship between the frequencies passed and the details / contrast in the final image

Fourier Optics





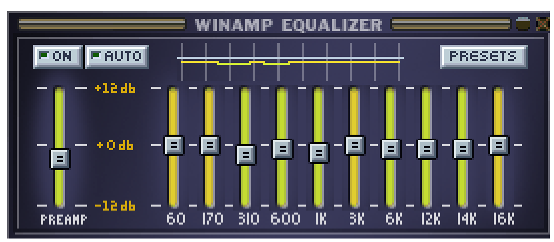
2D FT pairs → diffraction patterns (Using ImageJ)



FT is in Log(Abs) scale!! Why?

Time-domain: sound

- load your mp3
- plot time-series
- plot frequency amplitude, phase, power spectrum, linear/ log
- show frequency bands, i.e. “equalizer”
- adjust and play in real time



Equalizer from mp3 player

Space-domain: image

- load your image
- display image
- show 2D frequency amplitude, phase, power spectrum, linear/ log
- show rings of equal freq., “image equalizer”
- adjust and display in real time- example on next slide

Fourier Filtering (ImageJ)