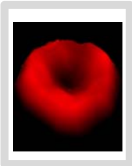

Chapter 1 – Math Toolbox

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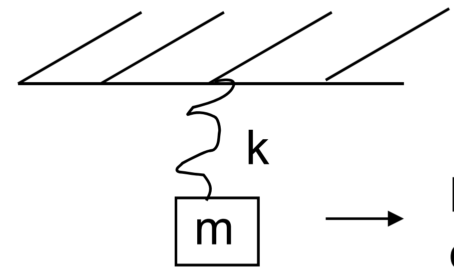
Objectives

- Develop a set of tools useful throughout the course

1.1 Linear Systems

- Consider a simple system:
- Equation of motion:



Mass on a spring

$$m\frac{d^2x}{dt^2}+\gamma m\frac{dx}{dt}+m\omega_o^{\,2}x = f(t) \qquad (1.1)$$

$$\omega_o = \sqrt{\frac{k}{m}}$$

- Define <u>Operator</u>: (linear differential eqs)

$$L = m\frac{d}{dt^2}+\gamma m\frac{d}{dt}+m\omega_o^{\,2} \qquad (1.2)$$

$$\rightarrow \boxed{L(x)=F(t)} \qquad (1.3)$$

1.1 Linear Systems

- Operator L has important properties:

a) $$L(ax) = m\frac{d^2(ax)}{dt^2} + \gamma m\frac{d(ax)}{dt} + m\omega_o^2(ax) =$$
$$= a\left[m\frac{d(x)}{dt^2} + \gamma m\frac{d(x)}{dt} + m\omega_o^2(x)\right] =$$
$$= aL(x) \qquad (1.4)$$

b) $$L(x+y) = m\frac{d(x+y)}{dt} + \gamma m\frac{d(x+y)}{dt} + m\omega_o^2(x+y) =$$
$$= L(x) + L(y) \qquad (1.5)$$

1.1 Linear Systems

- Definition: An operator obeying properties L(ax) = aL(x) and L(x+y)=L(x)+L(y) is called linear
- Most of the system in nature are linear; well, at least to the first approximation
- They are mathematically tractable → analytic solutions
- Consider equations:

$$\begin{cases} L(x_1) = 0 \\ L(x_2) = 0 \end{cases} \qquad (1.6)$$

→ x_1, x_2 are solutions

1.1 Linear Systems

- Continuing:

$\rightarrow L(ax_1 + bx_2) = L(ax_1) + L(bx_2)$

$= aL(x_1) + bL(x_2)$

$= 0 + 0$ (1.7)

- Any linear combination of solutions: x_1, x_2 is also a solution
- The number of independent solutions = degrees of freedom

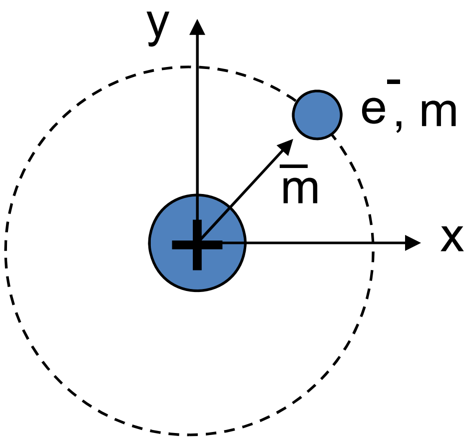
$X_1, X_2, \ldots, X_N$ = independent solutions if

$$X_i \neq \sum_{j \neq i} \alpha_j x_j \text{ , for any } \alpha_j \quad (1.8)$$

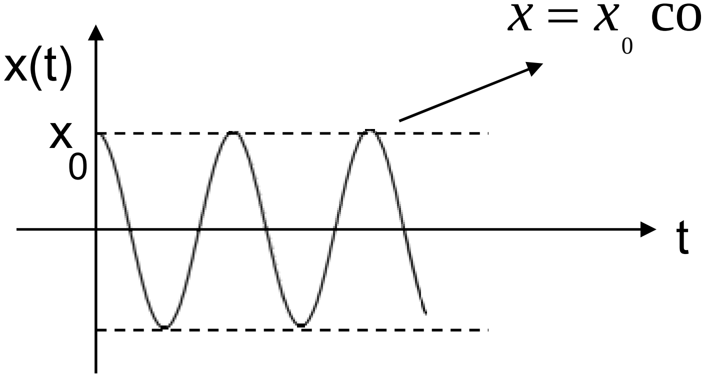
- Linear Differential eqs of order N allow for N independent solutions

1.2 Light-matter interaction

- Classic model of atom: e⁻ rotating around N ≈ planets

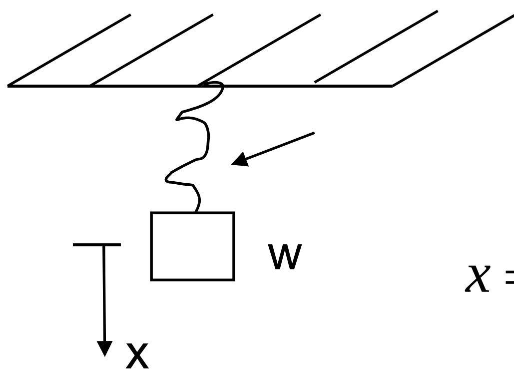


$$x = x_0 \cos(\omega_0 t)$$



→ Lorentz Model

- Analogy :



$$x = x_0 \cos(\omega_0 t) \quad \rightarrow \quad \omega_o = \sqrt{\frac{k}{m}}$$

1.2 Light-matter interaction

→ ▪ So, motion of charge follows the same eq (1.1)

$$m\frac{d^2x}{dt^2} + \gamma m\frac{dx}{dt} + m\omega_o{}^2 x = F(t)$$

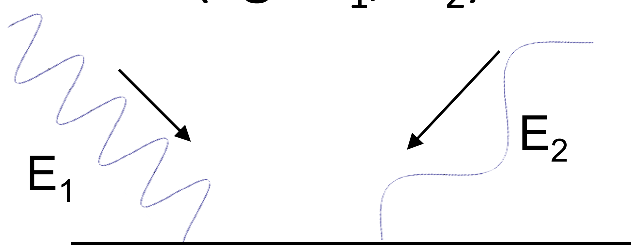
- Incident field drives the charge: $\bar{F}(t) = q\bar{E}(t)$ (1.9)
- For e^-, q = -e !
- Monochromatic field: $E(t) = E_o e^{-i\omega t}$

$$\rightarrow m\ddot{x} + \gamma m\dot{x} + m\omega_o{}^2 = qE_o e^{-i\omega t} \quad (1.10)$$

- This is the eq of motion for eletric charge under incident EM field. Can explain most of Optics!

1.3 Superposition principle

- Suppose we have 2 fields simultaneously interacting with the material (Eg. ω_1, ω_2):



$$E_1 = |E_1| e^{-i\omega_1 t} \;; qE_1 = F_1$$
$$E_2 = |E_2| e^{-i\omega_2 t} ; qE_2 = F_2 \qquad (1.11)$$

- Let x_1, x_2 be solutions of displacements for the two forces F_1 and F_2

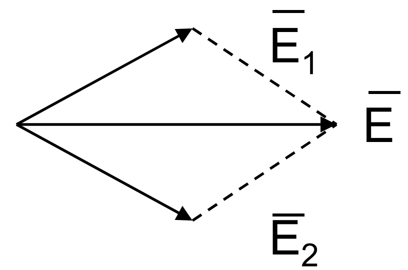
$$\begin{cases} L(x_1) = F_1(t) \\ L(x_2) = F_2(t) \end{cases} \qquad (1.12)$$

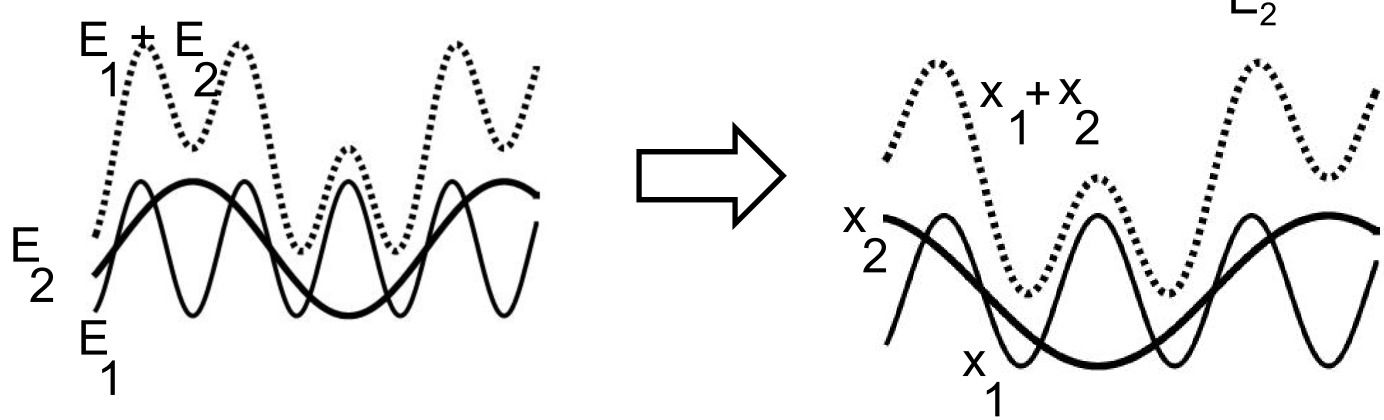
1.3 Superposition principle

- Consider the same solution:

$$\begin{aligned} L(x_1 + x_2) &= L(x_1) + L(x_2) \\ &= F_1(t) + F_2(t) \end{aligned} \tag{1.13}$$

- So, final solution is just the sum of individual solutions. Nice!
- This is the <u>superposition principle</u>
- For the 2 frequency example:

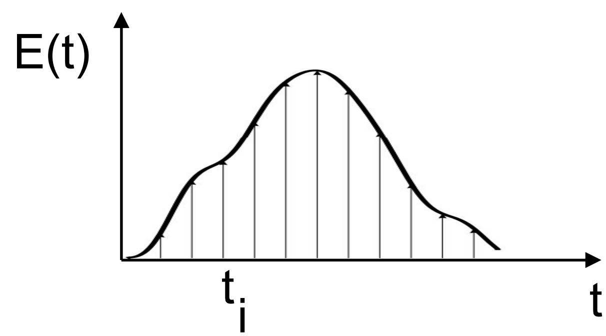




- It's as if one applies the fields one by one and sums their results

1.4 Green's function/impulse response

- Let the incident field, i.e driving field, have a complicated shape



→ arbitrary

- E(t) can be broken down into a succession of short pulses, i.e Dirac delta functions:

$$\rightarrow \delta(t) = \begin{cases} 1, \; t = 0 \\ 0, \; \text{otherwise} \end{cases} \qquad (1.14)$$

$$\rightarrow E(t) = \int_{-\infty}^{\infty} E(t')\delta(t-t')dt' \qquad (1.15)$$

1.4 Green's function/impulse response

- If we know the response of the system to a short pulse, $\delta(t)$, the problem is solved
- Let h(t) be the solution to $\delta(t)$
- The final solution for an arbitrary force $\bar{F}(t) = q\bar{E}(t)$ is:

$$x(t) = \int_{-\infty}^{\infty} E(t')h(t-t')dt' \qquad (1.16)$$

- This is the Green's method of solving linear problems
- h(t) = Green's function or impulse response of the system
- Complicated problems become easily tractable!

1.5 Fourier Transforms

- Very efficient tool for analyzing linear (and non-linear) processes
- Definition:

$$\Im[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-i2\pi x f_x} dx$$
$$= F(f_x) = \tilde{f}(\xi) \qquad (1.17)$$

- F is the Fourier transform of f

$$\boxed{(x, y, z) \xrightarrow{\Im} (\xi, \eta, \zeta)}$$

- $f : \Delta \rightarrow \Delta; \Delta \in \mathbb{C}$, f must satisfy:

  a) $\int |f| < \infty$ - modulus integrable

  b) f has finite number of discontinuities in the finite domain Δ

  c) f has no infinite discontinuities
- In practice, some of these conditions are sometimes relaxed

1.5 Fourier Transforms

- Inverse Fourier Transforms:

$$\Im^{-1}\left[\Im\left(f(x)\right)\right]=\int_{-\infty}^{\infty}\tilde{f}(\xi)e^{+i2\pi x f_x}df_x$$
$$=f(x) \qquad (1.18)$$

→ $\Im^{-1}[\Im(f)]=f$ (1.19)

- Meaning of F.T: reconstruct a complicated signal by summing sinusoidals with proper weighting

1.5 Fourier Transforms

- Fourier transform is a linear operator:

$$\begin{aligned}
&\Im[af(x)+bg(x)]= \\
&=\int_{-\infty}^{\infty}[af(x)+bg(x)]e^{-i2\pi x\xi}dx= \\
&=a\int_{-\infty}^{\infty}f(x)e^{-i2\pi x\xi}dx+b\int_{-\infty}^{\infty}g(x)e^{-i2\pi x\xi}dx \\
&=a\Im[f(x)]+b\Im[g(x)]
\end{aligned} \tag{1.20}$$

1.6 Basic Theorems with Fourier Transforms

a) Shift Theorem: if $\tilde{f}(\xi) = \Im[f(x)]$

$$\boxed{\Im\{f(x-a)\} = \tilde{f}(\xi)e^{-i2\pi\xi a}} \qquad (1.21)$$

- Easy to prove using definition
- Eq 1.21 suggest that a shift in one domain corresponds to a linear phase ramp in the other (Fourier) domain

1.6 Basic Theorems with Fourier Transforms

b) Parseval's theorem: if $\Im[f(x)] = \tilde{f}(\xi)$

$$\boxed{\int_{-\infty}^{\infty} \left| f(x) \right|^2 dx = \int_{-\infty}^{\infty} \left| \tilde{f}(\xi) \right|^2 d\xi} \tag{1.22}$$

- Conservation of total energy

1.6 Basic Theorems with Fourier Transforms

c) Similarity theorem: if

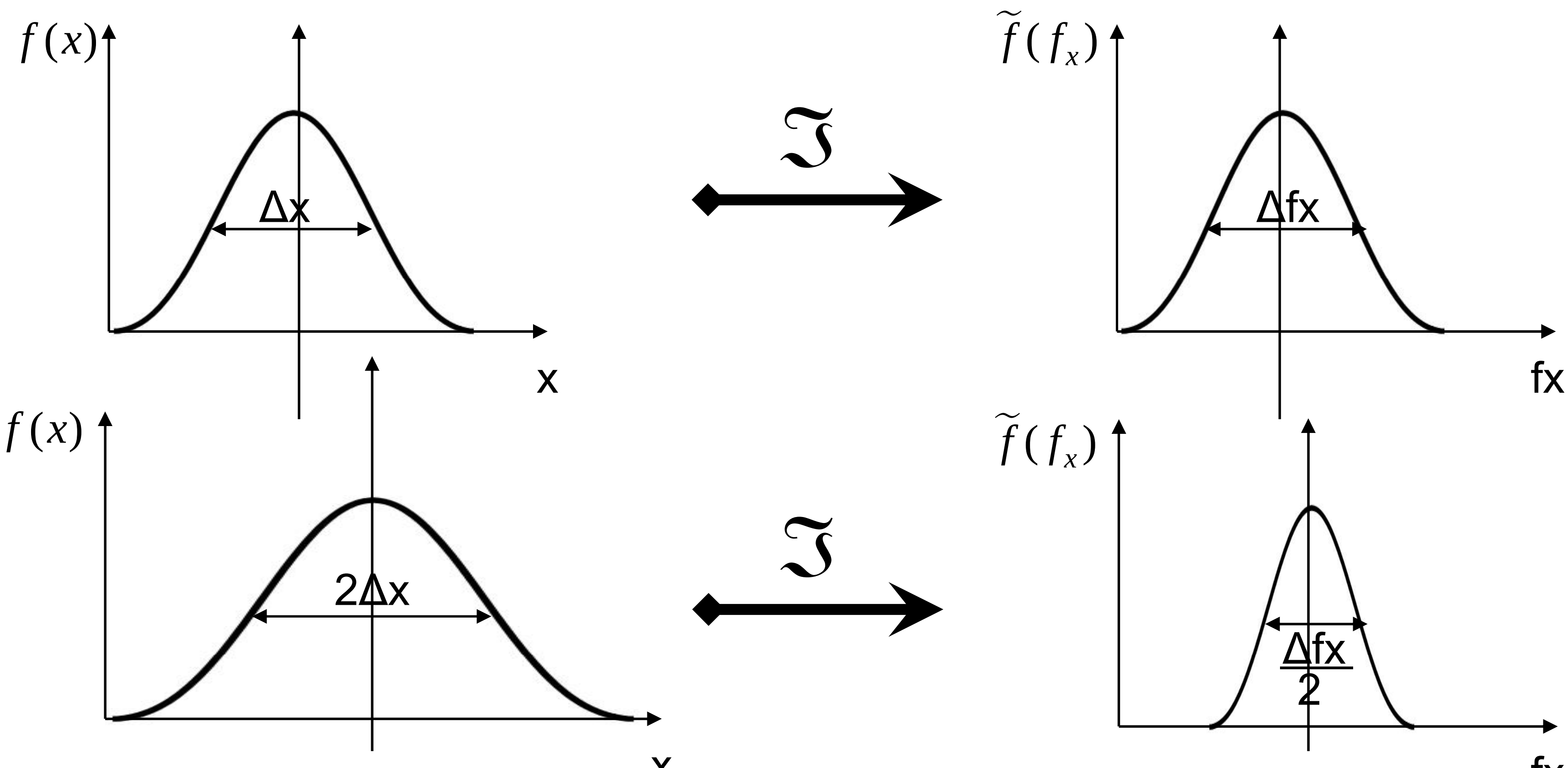
$\Im[f(x)] = \tilde{f}(f_x)$, i.e. $\tilde{f}$ is the F.T of f

$$\boxed{\Im[f(ax)] = \frac{1}{|a|}\tilde{f}\left(\frac{\xi}{a}\right)} \qquad (1.23)$$

- Theorem 1.23 provides intuitive feeling for F.T

1.6 Basic Theorems with Fourier Transforms

- Let's consider:

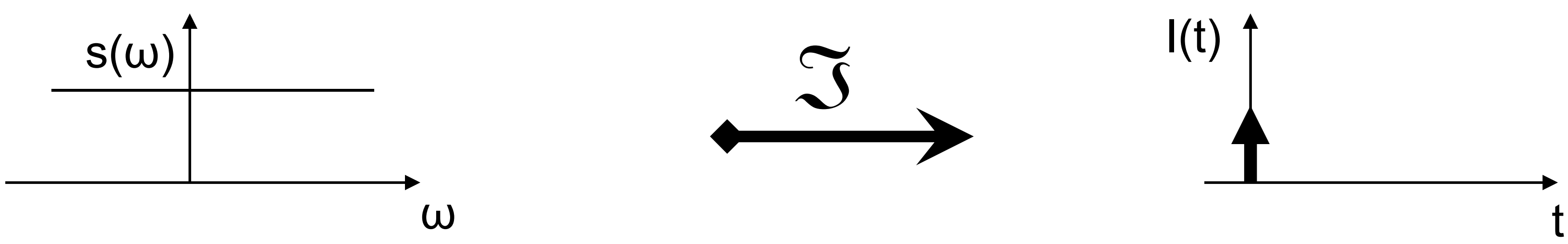


1.6 Basic Theorems with Fourier Transforms

! Broader functions in one domain imply narrower functions in the other and vice-versa

- Eg. To obtain short <u>temporal</u> pulses of light, one needs a broad spectrum (Ti: Saph laser)

! Only an infinite spectrum allows for δ -function pulses



→ Physically Impossible

1.6 Basic Theorems with Fourier Transforms

- Before we present the last theorems, we introduce the definitions of convolution and correlation
- Let

$$g(x) \xrightarrow{\Im} G(\xi)$$
$$h(x) \xrightarrow{\Im} H(\xi)$$

- Convolution of g and h:

$$g \circledv h = \int_{-\infty}^{\infty} g(x')h(x-x')dx' \quad (1.24)$$

- Correlation of g and h

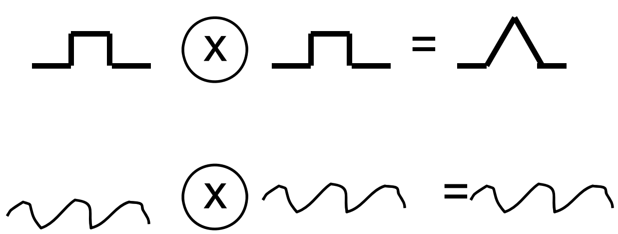
$$g \otimes h = \int_{-\infty}^{\infty} g(x')h(x'-x)dx' \quad (1.25)$$

1.6 Basic Theorems with Fourier Transforms

- Difference between $\otimes$ and ⓥ is h(x-x’) vs h(x’-x), i.e. flip vs non-flip of h
- Particular case:
  - Autocorrelation: g=h

$$g \otimes g = \int_{-\infty}^{\infty} g(x')g(x'-x)dx' \qquad (1.26)$$

- Exercise: Use PC to show:



Gauss X Gauss = Gauss

1.6 Basic Theorems with Fourier Transforms

d) Convolution theorem:

$$\Im[g \circledv h] = GH \tag{1.27}$$

i.e $$\Im\left[\int_{-\infty}^{\infty} g(x')h(x-x')dx'\right] = G(\xi)H(\xi)$$

- Convolution in one domain corresponds to a product in the other. Nice!
- Multiplication is always easy to do
- Recall Green's function: h(t) = the response to a δ-function light pulse

1.6 Basic Theorems with Fourier Transforms

- We found (Eq 1.16):

$$x(t) = \int_{-\infty}^{\infty} E(t^{'})h(t-t^{'})dt^{'}$$

i.e the response to an arbitrary field E(t) is the convolution

E ⓥ h!

- Let's take the F.T:

$$x(\omega) = E(\omega)h(\omega) \qquad (1.28)$$

→ It doesn't get any simpler than this

i.e if we know the impulse response h(t), (or the Green's function) take F.T → h(ω) ≡ <u>transfer function</u> → response to any field E is:

$$x(t) = \Im[E(\omega)h(\omega)] \qquad (1.29)$$

1.6 Basic Theorems with Fourier Transforms

e) Correlation theorem:

- $\otimes$ differs from ⓥ only by minus sign → similar theorem:

$$\Im[g \otimes h] = GH^* \tag{1.30}$$

i.e $$\Im[\int_{-\infty}^{\infty} g(x')h(x'-x)dx'] = G(\xi)H(\xi)^*$$

→ Particular case: g = h (auto correlation):

$$\Im[g \otimes g] = GG^* = |G|^2 \tag{1.31}$$

1.6 Basic Theorems with Fourier Transforms

- Eg: F.T of an auto correlation is the power spectrum
- Very important for both time and space fluctuating fields:

$$\begin{cases} \Gamma(t) = \int_{-\infty}^{\infty} E(t')E(t'-t)dt = \text{auto correlation} \\ \Im[\Gamma(t)] = E(\omega)E^{*}(\omega) = S(\omega) = \underline{\text{power spectrum}} \end{cases} \quad (1.32)$$

(Wiener–Khinchin theorem)

- We'll meet them again later!

1.7 Differential equations and Fourier Transforms

- Let f be a function of time:

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{+i\omega t} d\omega = \Im^{-1}(F) \tag{1.33}$$

- What is $\dfrac{\partial f}{\partial t}$?

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega\right] = \\ &= \int_{-\infty}^{\infty} F(\omega) \frac{\partial}{\partial t}\left[e^{i\omega t}\right] d\omega = \\ &= \int_{-\infty}^{\infty} [i\omega F(\omega)] e^{i\omega t} d\omega \\ &= \Im^{-1}\left[i\omega F\right]\end{aligned} \tag{1.34}$$

- So, $f \rightarrow F$ & $\partial f / \partial t \rightarrow i\omega F$ → Very Useful!

1.7 Differential equations and Fourier Transforms

- Great:

$$\begin{cases} \Im[f(t)] = F(\omega) \Rightarrow \text{Then} \\ \Im[\dfrac{\partial f(t)}{\partial t}] = i\omega F(\omega) \end{cases} \qquad (1.34)$$

- Now $\Im[\dfrac{\partial^2 f}{\partial t^2}] = \Im[\dfrac{\partial}{\partial t}(\dfrac{\partial f}{\partial t})] = i\omega i\omega F(\omega)$

$$= -\omega^2 F(\omega)$$

In others words: $\boxed{\Im[\dfrac{\partial^n f}{\partial t^n}] = i^n \omega^n F(\omega)}$ (1.35)

- Differentiation theorem

1.7 Differential equations and Fourier Transforms

- Why 1.35 result is important? Because linear differential equations are resolved in the frequency domain more easily
- Eg: Recall our e^- revolving around nucleus under field illumination E(t)

$$m\frac{d^2x(t)}{dt^2}+\gamma m\frac{dx(t)}{dt}+m\omega_o^2x(t)=qE(t) \qquad (1.36)$$

- The solution is x(t). But we can solve for x(ω)= $\Im[x(t)]$ and take $\Im^{-1}$ in the end

1.7 Differential equations and Fourier Transforms

- So, let's take F.T of 1.36, using the differentiation theorem:

$$m[-\omega^2 x(\omega)] + i\omega\gamma m x(\omega) + m\omega_o^{\,2} x(\omega) = qE(\omega)$$

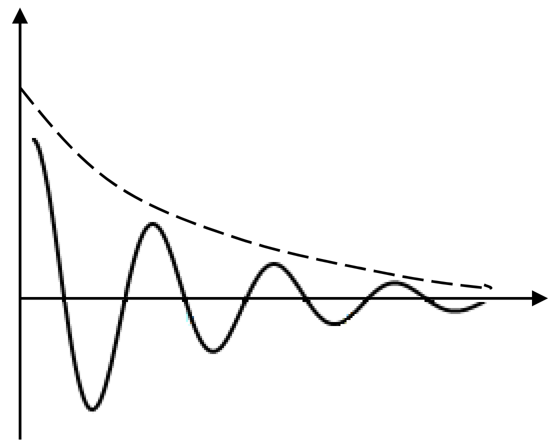
$$x(\omega)[-m\omega^2 + i\omega\gamma m + m\omega_o^{\,2}] = qE(\omega)$$

Since q=-e:

$$\boxed{x(\omega) = \frac{\frac{e}{m}E(\omega)}{\omega^2 - i\gamma\omega - \omega_o^{\,2}}} \qquad (1.37)$$

1.7 Differential equations and Fourier Transforms

- Exercise: use PC to take $\Im^{-1}$ of Eq. 1.37



- "damped" oscilation, γ = damping factor
  ! Problem solved

1.8 Refraction and absorption. Dispersion

- Given the electron displacement as a function of frequency, x(ω), we can define the dipole moment:

$$\bar{p} = q\bar{x}$$

$$\boxed{p = -ex} \qquad (1.38)$$

- The dipole moment is a microscopic quantity; we need a macroscopic counterpart:

$$\bar{P} = N\left\langle \bar{p} \right\rangle = \frac{-\dfrac{Ne^2}{m}E}{\omega^2 - \omega_o^{\ 2} - i\gamma\omega} \qquad (1.39)$$

1.8 Refraction and absorption. Dispersion

- $\bar{P} \equiv$ induced polarization
- $\bar{N} \equiv$ concentration [m^{-3}]
- But $\bar{P}$ relates to the macroscopic response of the material χ, i.e. eletric susceptibility:

$$\bar{P} = \varepsilon_o \chi \bar{E} \tag{1.40}$$

- ε_o = permeability of vacuum
- Finally, $\chi = \varepsilon_r - 1 = n^2 - 1$ (1.41)
- ε_r = relative permeability
- n = refractive index
- If $\chi \in \mathbb{R}$, as opposed to $\bar{\bar{\chi}} = \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix}$, material is isotropic

1.8 Refraction and absorption. Dispersion

- So, combining 1.39 and 1.40:

$$\chi = \frac{Ne^2 / m\varepsilon_o}{(\omega^2 - \omega_o^{\ 2}) - i\gamma\omega} = n^2 - 1 \in \mathbb{C} \tag{1.42}$$

- For low-n materials, such as rarefied gases,

$$n^2 - 1 = (n-1)(n+1) \simeq 2(n-1)$$

$$\rightarrow \quad n = 1 + \frac{Ne^2}{2m\varepsilon_o} \frac{1}{(\omega^2 - \omega_o^{\ 2}) - i\gamma\omega}$$

$$= n' + in'' \tag{1.43}$$

1.8 Refraction and absorption. Dispersion

$$\rightarrow \begin{cases} n' = 1 + \dfrac{Ne^2}{2m\varepsilon_o} \dfrac{\omega^2 - \omega_o{}^2}{(\omega^2 - \omega_o{}^2) - \gamma^2\omega^2} & (1.44a) \\ n'' = \dfrac{Ne^2}{2m\varepsilon_o} \dfrac{\gamma\omega}{(\omega^2 - \omega_o{}^2) - \gamma^2\omega^2} & (1.44b) \end{cases}$$

- n’ = Re(n) = refractive index
- n” = Im(n) = absorption index

1.8 Refraction and absorption. Dispersion

- Eg: Plane wave:

$$E = E_o e^{i\bar{k}\cdot\bar{r}};k = nk_o$$

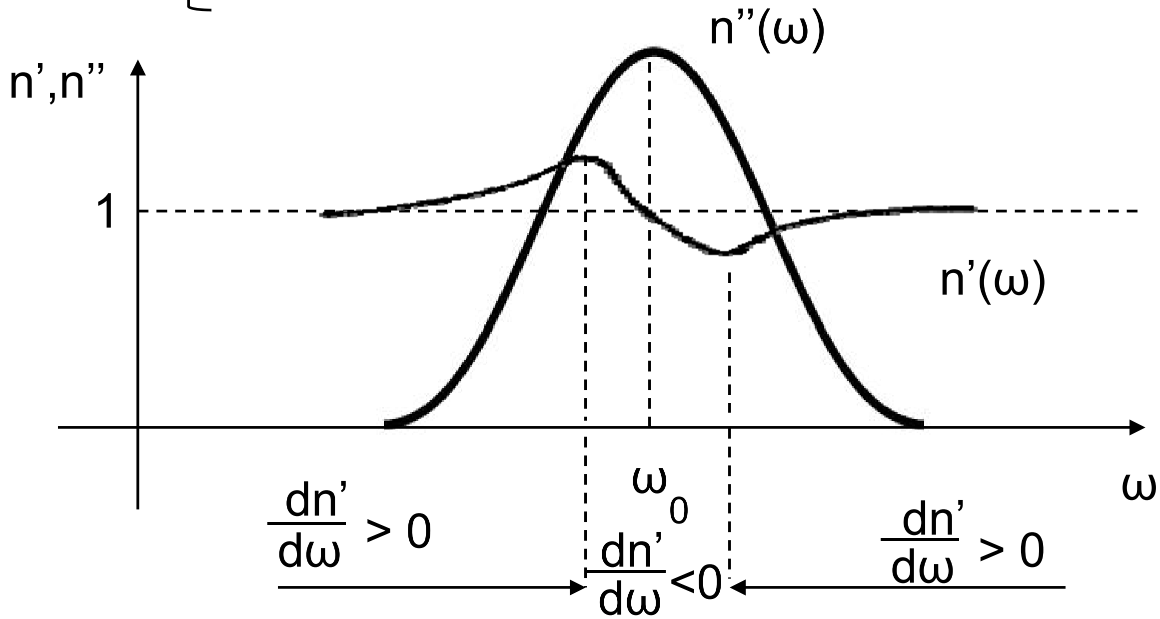
$$E = E_o e^{ink_o r} =$$
$$= E_o e^{ik_o r(n' + in'')}$$

$$\boxed{E = E_o \underbrace{e^{-n''k_o r}}_{\text{absorption}} \underbrace{e^{in'k_o r}}_{\text{refraction}}}$$

$\alpha = n''k_o =$ absorption coefficient

1.8 Refraction and absorption. Dispersion

- Definition:
  - $n'(\omega)$ = variation of refractive index with frequency = dispersion
  - $n''(\omega)$ = absorption line shape



1.8 Refraction and absorption. Dispersion

- Note the line shape:

$$\frac{\gamma\omega}{(\omega^2-\omega_o{}^2)+\gamma^2\omega^2} \simeq \frac{1}{\gamma\omega}\frac{1}{1+\left(\frac{(\omega-\omega_o)2\omega_o}{\gamma\omega}\right)^2} = \frac{1}{\gamma\omega}\frac{1}{1+\left(\frac{\omega-\omega_o}{\gamma\omega/2\omega_o}\right)^2}$$

- Lorentz function:

$$\Im(\omega)=\frac{1}{a}\left(\frac{1}{1+(\omega/a)^2}\right)$$; a = width

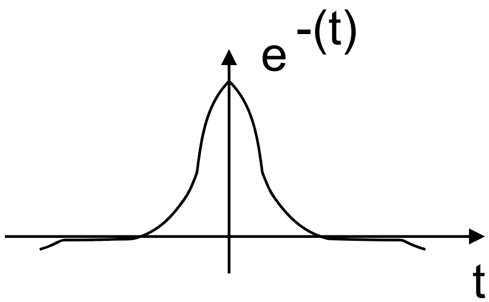
1.8 Refraction and absorption. Dispersion

- Thus the asorption line is a Lorentzian:

$$\alpha(\omega;\omega_o) = \frac{1}{\Delta\omega}\frac{1}{1+\left(\frac{\omega-\omega_o}{\Delta\omega}\right)^2};\Delta\omega \sim \gamma$$

$$\boxed{\Im[\alpha(\omega)] = e^{-\Delta\omega|t|}}$$

$\rightarrow$



- The Fourier transform of a Lorentzian is an exponential!

1.8 Refraction and absorption. Dispersion

Connect to quantum mechanics:

- 2 level system:

  1 → E_1

  0 → E_o

$$\Delta E = E_1 - E_o = \hbar(\omega_1 - \omega_o)$$

- Probability of spontaneous emission/absorption:
  - $p(t) \sim e^{-t/t_{lifetime}}$ → exponential decay
  - Linewidth is Lorentz = natural linewidth

! The model of e^- on springs was introduced by... Lorentz

1.9 Maxwell's Equations

- Fully describe the propagation of EM fields
- Quantify how $\hat{E}$ and $\hat{H}$ generate each other

$$\begin{cases} \nabla \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t} & \text{I} \\ \nabla \times \overline{H} = \dfrac{\partial \overline{D}}{\partial t} + \overline{j} & \text{II} \\ \nabla \overline{D} = \rho & \text{III} \\ \nabla \overline{B} = 0 & \text{IV} \end{cases} \qquad (1.45a)$$

1.9 Maxwell's Equations

- Plus material equations

$$\begin{cases} \overline{D} = \varepsilon_o \overline{E} + \overline{P} & \text{V} \\ \quad = \varepsilon \overline{E} & \\ \overline{B} = \mu_o \overline{H} + M & \text{VI} \end{cases}$$

- Definitions:

$\overline{E} =$ Eletric field vectors $\quad \rho =$ charge density

$\overline{H} =$ Magnetic field vectors $\quad \overline{j} = \sigma \overline{E}$ = current density

$\overline{D} =$ Eletric displacement $\quad \overline{P} =$ polarization

$\overline{B} =$ Magnetic inductance $\quad \overline{M} =$ magnetization

1.9 Maxwell's Equations

- Let's combine I and II (assume no free charge: ρ= 0, $\bar{j}$ =0)
- <u>Use property</u>: $\nabla\times(\nabla\times\bar{E})=\nabla(\nabla\bar{E})-\nabla^2\bar{E}$
- Since $\rho=0\Rightarrow\nabla\bar{E}=0\Rightarrow\boxed{\nabla\times(\nabla\times\bar{E})=-\nabla^2\bar{E}}$
- Take $\nabla\times(EqI)$:

$$\nabla\times(\nabla\times\bar{E})=-\nabla\times\left(\frac{\partial\bar{B}}{\partial t}\right) \quad (1.46)$$

$$\rightarrow -\nabla^2\bar{E}=\frac{\partial}{\partial t}(\nabla\times\bar{B})=$$

$$=-\frac{\partial}{\partial t}(\mu_o\nabla\times\bar{H})=$$

(see next slide)

$$=-\frac{\partial}{\partial t}(\mu_o\frac{\partial\bar{\Delta}}{\partial t})=$$

1.9 Maxwell's Equations

$$\rightarrow \quad -\nabla^2 \bar{E} = \frac{\partial}{\partial t}(\nabla \times \bar{B}) =$$

$$= -\frac{\partial}{\partial t}(\mu_o \nabla \times \bar{H}) =$$

$$= -\frac{\partial}{\partial t}(\mu_o \frac{\partial \bar{\Delta}}{\partial t}) =$$

$$= -\varepsilon\mu \frac{\partial^2 E}{\partial t^2}$$

Thus: $$\boxed{\nabla^2 \bar{E} - \varepsilon\mu \frac{\partial^2 E}{\partial t^2} = 0} \qquad (1.47)$$

- Wave Equation

1.9 Maxwell's Equations

Note:

$$\begin{cases} \varepsilon\mu = \dfrac{1}{v^2}; \qquad v = \dfrac{c}{n}; \\ n = \sqrt{\dfrac{\mu\varepsilon}{\mu_o\varepsilon_o}} \end{cases}$$

c = speed of light in vacuum

- The wave equation describes the propagation of a atime-dependent field (eg. pulse)
- <u>Solution</u>: plane wave: $E = E_o e^{-i(\omega t - \bar{k}\cdot\bar{r})}$
  - $k = \dfrac{2\pi}{\lambda} = \dfrac{\omega}{v} = n\dfrac{\omega}{c}$; k = wave equation

1.9 Maxwell's Equations

- Phase of the field:

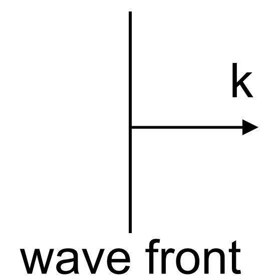
$$\varphi = \omega t - \bar{k}\cdot\bar{r} \qquad (1.48)$$

- Note: φ = constant describes a surface that moves with a certain velocity

$\boxed{\omega t - \bar{k}\cdot\bar{r} = \text{constant}}$ eq of planes $\perp \bar{k}$

$\rightarrow \omega dt - kdr = 0$

$$\rightarrow \boxed{\frac{dr}{dt} = \frac{\omega}{k} = v_p} \qquad (1.49)$$



The surface of constant phase is traveling with velocity:

$v_p = \dfrac{\omega}{k} =$ phase velocity

1.9 Maxwell's Equations

- What is the counterpart of the wave equation in the frequency domain?
- Well, remember $\frac{\partial}{\partial t} \xrightarrow{\Im} i\omega$
- Upon Fourier transforming, Eq. 1.47 becomes:

$$\nabla^2 \bar{E} - \frac{1}{v^2}(i\omega . i\omega) E(\omega) = 0$$

$$\rightarrow \quad \nabla^2 \bar{E} + \frac{1}{v^2}(\omega^2) E(\omega) = 0$$

- Note: $k = \frac{\omega}{v}$

$$\rightarrow \quad \nabla^2 E(\omega) + k^2 E(\omega) = 0 \qquad (1.50)$$

1.9 Maxwell's Equations

→ $\nabla^2 E(\omega) + k^2 E(\omega) = 0$

- The equation above is the "Helmholtz equation"
- Describes how each frequency ω propagates