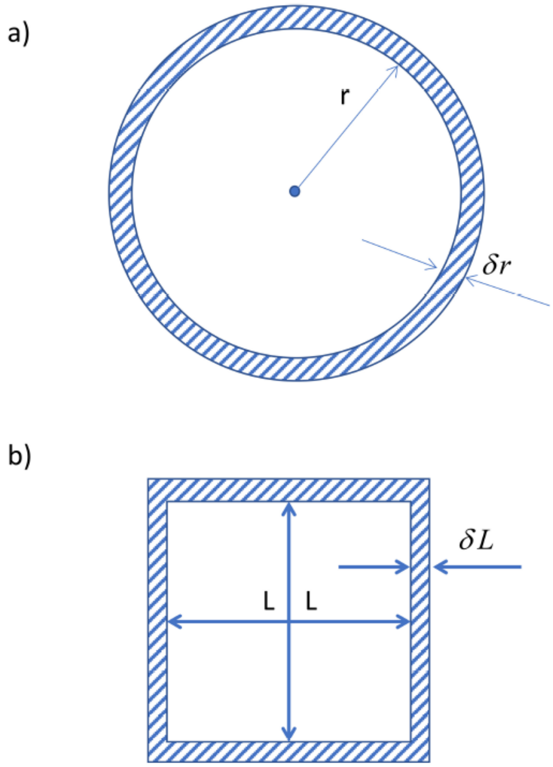
ECE 460: Optical Imaging

Homework1: Fourier Transform

Due 01/28/2020

1. Consider the input to an LSI system to be the step function, $\Gamma(x)$. Prove that the derivative of the output, g(x), yields the impulse response of the system, namely,
$$\frac{dg(x)}{dx} = h(x)$$
2. Prove the following property of the derivative of the δ function.
$$\int_{-\infty}^{\infty} f(x)\delta'(x-b)dx = -f'(b)$$
Where f is an arbitrary function, derivable at x=b.
3. Evaluate the following integral in terms of its Fourier Transform, $\tilde{f}(\omega)$
$$\int_{-\infty}^{t} f(t')dt'$$
4. Evaluate the following integrals using properties of FT
   a) $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$
   b) $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx$
   c) $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^4 dx$
5. Prove the following Fourier transform pair
$$F(t) = \Gamma(t)e^{-at}\sin(\omega_0 t)$$
$$F(\omega) = \frac{\omega_0}{\omega_0^2 + (a+i\omega)^2}$$
6. Prove that the autocorrelation of the derivative of a function is proportional to the second order derivative of the autocorrelation of the function, meaning
$$\frac{\partial f(t)}{\partial t} \otimes \frac{\partial f(t)}{\partial t} = -\frac{\partial^2[f \otimes f]}{\partial t^2}$$
7. The 1D wave equation for an impulse source both in time and space is
$$\frac{\partial^2 h(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 h(x,t)}{\partial t^2} = \delta(x)\delta(t)$$
Solve the equation in the spatiotemporal frequency domain, $\tilde{h}(k_x, \omega)$
8. Compute the convolution
$$f(\mathbf{r}_\perp) \circledv e^{i\mathbf{k}_\perp \cdot \mathbf{r}_\perp},$$
Where $r_\perp = (x, y), k_\perp = (k_x, k_y)$
9. Compute the 2D Fourier transforms of the functions illustrated below (shaded area has a value of 1 and the rest of 0).



. (a) Ring of radius r and thickness δr. (b) Frame of side L and thickness δL.

10. Solve the 2D diffusion equation in the spatiotemporal frequency domain, i.e. find $h(k_\perp, \omega)$ for $r_\perp = (x, y)$.

$$D\nabla_\perp^2 h(r_\perp, t) - \frac{\partial h(r_\perp, t)}{\partial t} = \delta(\mathrm{r}_\perp)\delta(t)$$

11. For the 3D wave propagation in vacuum, (equation given below), solve for $h(r, \omega)$

$$\nabla^2 h(r, \omega) + k_0^2 h(r, \omega) = \delta^3(r)$$